ON SOME PROPERTIES OF WING GRAPHS

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Abstract: The study of Graph operators began with Harary's, Norman's and Ore's study on 'Line Graphs' in the 1960s and it has multiplied to a vast area today. In this article, we study some properties of a special graph operator known as the ' Wing Graph'.

Key Words: Graph, Wing Graph, radius of a graph, the Complete Graph.

1. Introduction:

Definition 1.1: A **graph** G = (V, E) is a pair consisting of some finite set 'V', the 'vertex set' and some subset 'E' of the set of all two element subsets of V, the ' edge set'.

We only consider graphs without multiple edges and loops. If there is an edge between vertices x and y, then the vertices are called **adjacent**.

Definition 1.2: A **walk** in a graph G is a sequence of vertices v_0, v_1, v_2,………v_l such that the v_i s are distinct and v_i and v_{i+1} are adjacent for every 'i' in {0, 1, 2, ………….., l-1}. The number 'l' is called the **length** of the path. The path is called a cycle if $v_0 = v_l$ and it is denoted by C_{l+1}. A graph is said to be **connected** if there is a path between any pair of vertices in it.

Definition 1.3: The **distance $d_G(x, y)$** of two vertices 'x' and 'y' in a connected graph G is the length of a shortest path between them.

Definition 1.4: The **eccentricity** of a vertex 'x' of a graph G is given by $e_G(x) = \text{Sup}\{d_G(x, y)/y \varepsilon V(G)\}$.The **radius r(G)** is the minimum eccentricity of a vertex in G and the **diameter d(G)** is the supremum of the set of all eccentrities appearing in G.

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Definition 1.5: A **graph dynamical system** is a pair (Γ, Φ) where Γ is a set of graphs and Φ : Γ →Γ is a mapping called an 'operator'.

Definition 1.6: The **complete bipartite** graph $K_{m,n}$ has a partition of the vertex set into two sets V_1 and V_2 of cardinality m and n respectively with vertices x and y adjacent whenever x is in V_1 and y is in V_2 and vice versa.

In this paper, some results on a graph operator known as the ' wing graph' are presented. It is defined as follows:

Definition 1.7: The **Wing Graph** W(G) of a graph G has all its edges as vertices . Two edges of G are adjacent vertices in W(G) if they are non incident edges of some induced 4-vertex path in G.

For example, $W(C_{2k+1}) = C_{2k+1}$

Wing Graphs were primarily studied in the quest to crack the Perfect Graph Conjecture. They can, however, be studied for several interesting properties of their own.

$W(C_{2n})$, $n \geq 3$, consists of exactly two disconnected components, each of which is C_n . Also, if a graph G is of radius $r(G) \leq 2$, then $d(u, v) \leq 2$ for every u, v in V(G). Then there is no induced P_4 in G. Such a G will have W(G) with no edges.

In all the following results, we assume that G is connected with r(G)>2. The terms and definitions not defined here are taken from (3).

2. Some properties of the Wing Graph:

Theorem 1: For any graph G, W(G) will never be isomorphic to $K_{m,n}$

Proof:

Suppose W(G) is isomorphic to $K_{m,n}$ .Let the two sets of vertices of $K_{m,n}$ be $v_0, v_1, v_2, \ldots\ldots\ldots v_m$ and $w_0, w_1, w_2, \ldots\ldots\ldots w_n$. Consider v_1 adjacent to w_1. Then they correspond to non adjacent edges of an induced P_4 in G. Let 'e' be the third edge in that P_4. Then 'e' cannot be any of the v_is , $i \neq 1$, since that would make it adjacent to

w_1 in W(G), which is not possible. Similarly, 'e' cannot be any of the w_is , $i \neq 1$. This means e cannot be in either set of the bipartition, a contradiction. Hence the theorem.

Remark 1:

The wing graph is non isomorphic to the complete graph on 2 or more vertices. This is because two adjacent edges in G can never be adjacent in W(G).

Theorem 2:

Given any connected graph H, there exists a connected graph G such that H is an induced subgraph of W(G).

Proof:

Let the vertices of H be $v_0, v_1, v_2, \ldots\ldots\ldots v_n$. Consider the graph G constructed as follows: To each vertex v_i in H, attach a pendant vertex w_i , forming an edge e_i.

Claim: The subgraph G' of W(G) induced by $e_1, e_2, \ldots\ldots.., e_n$ is isomorphic to H.

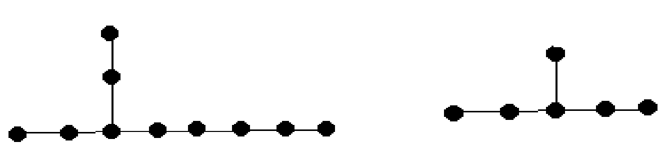
$|G'| = n = |H|$. e_i adjacent to e_j in G' implies that they are non adjacent edges of an induced P_4 in G. This in turn implies that v_i is adjacent to v_j in H. Conversely, if $v_i - v_j$ is an edge in H, then $w_i - v_i - v_j - w_j$ is an induced P_4 in G. This implies $e_i – e_j$ is an edge in G', where e_i is $w_i - v_i$ and e_j is $v_j – w_j$. Thus G' is isomorphic to H.

Remark 2:

The above result shows that there does not exist a forbidden subgraph characterization for wing graphs.

Remark 3:

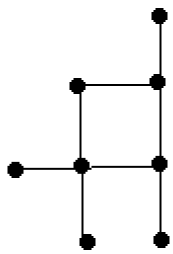
G' in the above theorem is not a unique construction. For example, both the following graphs have wing graphs with induced P_5 :



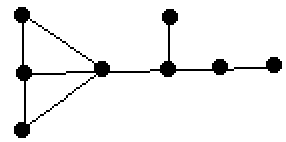
Remark 4: For any graph G, the number of induced P_4 s in G is the same as the number of induced P_4 s in the complement of G. So the order of W(G) and order of $W(G^C)$ are the same.

Remark 5: G is H – free does not imply that W(G) is H – free.

For example, consider the triangle free graph G whose wing graph has induced triangles :



G



W(G)

Theorem 3: Given any number ‘a’>1, there exists G such that r(G) = r[W(G)] = a.

Proof: Consider the odd cycle C_{2a+1}.Construct G as follows: Introduce a vertex ‘v’ adjacent to two adjacent vertices of $C_{2a+1.}$ Then r(G) = a. Now, W(G) is C_{2a+1} with two pendant vertices attached to the end vertices of an induced P_3. Then r[W(G)] = a.

Theorem 4: If W(G) is isomorphic to P_n , then n = 5, 6 or 7. In other words, if P_n is the wing graph of some graph G, then it is P_5, P_6 or P_7 .

Proof: Suppose W(G) is isomorphic to $P_{n\,.}$ Obviously, it has to be connected. Further, G must not contain any induced cycles, so that W(G) is cycle free. Hence G is a tree on n+1 vertices. Further, every vertex ‘u’ in G must be such that e(u) ≥3. This makes sure there are no isolated vertices in W(G).

Now, when e(u) = 3 , deg(u) > 3 would imply the existence of a vertex in W(G) of degree ≥3, which means W(G) is not a path. So e(u) = 3 implies deg(u) ≤ 3.

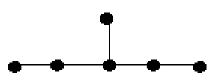
Similarly, when e(u) > 3, deg(u) > 2 would imply the existence of a vertex in W(G) of degree ≥3, which means W(G) is not a path. So e(u) > 3 implies deg(u) ≤ 2.

So all vertices in G are of degree 1, 2 or 3 only. Next, we see that if there were no vertices of degree 3 in G, G would be a path and hence W(G) would be disconnected. So there is at least one vertex of degree 3 in G. Next, if there were more than one vertex of degree 3 in G, say u and v, , there would be a unique path between them. This would lead to a disconnection in W(G). Hence there is exactly one vertex of degree 3 in G.. The remaining n vertices are of degree 2 or 1.

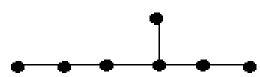
Let n_1 be the number of vertices of degree 1 in G and n_2 be the number of vertices of degree 2 in G. Then by the Fundamental Theorem of Graph Theory,

$n_1 + n_2 + 3 = 2n$ and $n_1 + n_2 = n$. Solving these equations, $n_1 = 3$ and $n_2 = n - 3$.

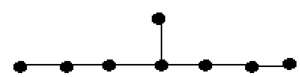
We see that G is a tree on n+1 vertices with 3 pendant vertices, one vertex of degree 3 and n-3 vertices of degree 2. So G can be constructed for n = 5, 6, and 7 , keeping W(G) a path, as follows:



G_1



G_2



G_3

These graphs have wing graphs P_5, P_6 and P_7 respectively. For n = 9, onwards, there appears an edge lying in 3 or more induced P_4s . This edge corresponds to a vertex of degree ≥ 3 in W(G), preventing it from being a path. Hence the only possible values of n are 5, 6 or 7.

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