LIE ALGEBRA

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By

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CERTIFICATE

This is to certificate that the project entitled “ LIE ALGEBRA is a bonafide record of the studies undertaken by MARY ALEETA P.J (Reg no: 180011015184), in partial fulfillments of the requirements for the award of M.Sc. Degree in mathematics at Department of Mathematics, St.Paul’s College, Kalamassery, during 2018-2020.

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Examiner :

DECLARATION

I MARY ALEETA P.J declare that, this project titled "LIE ALGEBRA" has been prepared by me under the supervision of Ms. MAYA.K , Jr. lecturer in Department of Mathematics, St.Paul's College, Kalamassery.

I also declare that this project has not been submitted by me fully or partially for the awards of any degree, diploma, title or recognition earlier.

Date: **MARY ALEETA P.J**

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MARY ALEETA P.J

LIE ALGEBRA

CONTENT

INTRODUCTION

Lie algebra is a vector space endowed with a special non-associative multiplication called a Lie bracket. It arises naturally in the study of mathematical objects called Lie groups, which serve as groups of transformations on spaces with certain symmetries. Hermann Weyl introduced the term “Lie algebra” (after Sophus Lie) in the 1930s. In older texts, the name “infinitesimal group” is used.

Lie algebra was named after Sophus Lie (pronounced “lee”), a Norwegian mathematician who lived in the latter half of the 19^{th} century. He studied continuous symmetries of geometric objects (i.e., Lie groups)called manifolds, and their derivatives (i.e., the elements of their Lie algebras).

Lie algebra theory is a rich and beautiful subject for those students of physics and mathematics who wish to study the structure of objects and expects to pursue further studies in geometry, algebra or analysis.

We start with some basic concepts in Lie algebra including ideals, homomorphisms, abstract Lie algebra and Lie algebra of derivations . Lie algebras are closely related to Lie groups which are groups of symmetries with a topological structure. So here we discuss some examples of Lie groups and their related Lie algebras. The first chapter also including the solvable and nilpotent Lie algebras.

The second chapter brings the idea about the semisimple Lie algebras including Killing form and complete reducibility of representations. We discuss about the root space decomposition of a semisimple Lie algebra in the third chapter and about the Root system in the fourth chapter.

Lie algebra play a fundamental role in modern mathematics and physics. Since they have a wide range of applications in different fields only few of its applications are discussing in the last chapter.

Chapter 1

Introduction to Lie algebra

1.1. Basic Concepts

Definition 1.1.1

A **Lie algebra** L is a vector space over a field F with a binary operation [,]: $L \times L \to L$ called Lie bracket or commutator , which satisfies,

L_1 : $[\alpha x + \beta y, z] = \alpha [x, z] + \beta [y, z]$ and

$[x, \alpha y + \beta z] = \alpha [x, y] + \beta [x, z]$, for all $x, y \in L$ and for all $\alpha, \beta \in$ F

(Bilinearity)

L_2 : $[x, x] = 0$, for all $x \in L$ (Antisymmetry)

L_3 : $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for all $x, y, z \in L$

(Jacobi identity)

A Lie algebra is called real or complex when the vector space is respectively real or complex .

Note 1.1.1

For any $x, y \in L$,

$[x+y, x+y] = [x, x] + [x, y] + [y, x] + [y, y]$ .... (1)

But by L_2 we have ,

$[x, x] = 0$, $[y, y] = 0$, $[x+y, x+y] = 0$

Therefore (1) implies , $[x, y] = -[y, x]$ for all $x, y \in L$ ....(L_2')

Thus, Lie bracket is anticommutative.

Definition 1.1.2

Let L and M be Lie algebras and $\phi : L \to M$ a bijection such that for all $\alpha, \beta \in F$,

$\phi(\alpha x + \beta y) = \alpha\,\phi(x) + \beta\,\phi(y)$

$\phi([x, y]) = [\phi(x), \phi(y)]$ then, ϕ is called an **isomorphism** and the Lie algebra L and M are called isomorphic .

Definition 1.1.3

A Lie algebra L is called **abelian** if $[x, y] = 0$ for all $x, y \in L$

Definition 1.1.4

A subset K of a Lie algebra L is called a **subalgebra of L** if , for all $x, y \in K$ and for all $\alpha, \beta \in F$

a) $\alpha x + \beta y \in K$
b) $[x, y] \in K$

Definition 1.1.5

A **Lie group** is an algebraic group $(G, *)$ that is also a smooth manifold such that

1. The inverse map $g \mapsto g^{-1}$ is a smooth map $G \to G$
2. The group operation $(g, h) \mapsto g * h$ is a smooth map $G \times G \to G$.

Example 1.1.1

The **general linear group** over the real numbers , denoted by $GL(n, R)$ is the group of all $n \times n$ invertible matrices with real number entries. We can similarly define it over the complex numbers ,C denoted by $GL(n, C)$.

Let V is a finite dimensional vector space over F. Let End V denote set of all linear transformations from V to V. End V is a vector space over F with dimension n^2 and it is a ring relative to the usual product operation. Then End V with the operation $[x, y] = xy - yx$ called the bracket of x and y is a Lie algebra over F. This Lie algebra is called the **general linear algebra** denoted by $\mathfrak{gl}(V)$ and it can also be identified with the set of all $n \times n$ matrices over F , denoted by $\mathfrak{gl}(n,F)$

Example 1.1.2

The **special linear groups** ($SL(n,C)$ and $SL(n,R)$) is the group of $n \times n$ invertible matrices having determinant 1. Since determinant is a continuous function, if a sequence A_n in $SL(n,C)$ converges to A, then A also has a determinant 1 and A $\in SL(n,C)$

Let dim $V = \ell+1$. Denoted by $\mathfrak{sl}(V)$ or $\mathfrak{sl}(\ell+1,F)$, the set of endomorphisms of V having trace zero.

Since, $Tr(xy) = Tr(yx)$ and

$Tr(x+y) = Tr(x) + Tr(y)$,

$\mathfrak{sl}(V)$ is a subalgebra of $\mathfrak{gl}(V)$, called the **Special linear algebra**.

Example 1.1.3

Let dim $V = 2\ell$,with basis $(v_1,v_2,...,v_{2\ell})$.Define a nondegenerate skew – symmetric form f on V by the matrix $S = \begin{pmatrix} 0 & I\ell \\ -I\ell & 0 \end{pmatrix}$. The set of all endomorphisms x of V satisfying $f(x(v), w) = -f(v, x(w))$ is called the **Symplectic algebra** denoted by $\mathfrak{sp}(V)$ or $\mathfrak{sp}(2\ell,F)$. dim $\mathfrak{sp}(2\ell,F) = 2\ell^2 + \ell$.

Example 1.1.4

An $n \times n$ matrix A is **orthogonal** denoted by $O(n)$ if the column vectors that make up A are orthonormal, that is ,if

$$\sum_{i=1}^{n} A_{ij} A_{ik} = \delta_{jk}$$

Equivalently, A is orthogonal if it preserves inner product, namely if

$< x, y > = < Ax, Ay >$ for all $x, y \in \mathbb{R}^n$.

Let dim $V = 2\ell + 1$ be odd and take f to be the nondegenerate symmetric bilinear form on V whose matrix is $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I\ell \\ 0 & I\ell & 0 \end{pmatrix}$. The set of all endomorphisms of V satisfying $f(x(v), w) = -f(v, x(w))$ is called the **Orthogonal algebra** $\mathfrak{o}(V)$ or $\mathfrak{o}(2\ell + 1, F)$. dim $\mathfrak{o}(V) = 2\ell^2 + \ell$.

Similar to $SL(n,\mathbb{C})$, the **special orthogonal group**, denoted by $SO(n)$, is defined as subgroup of $O(n)$ whose matrices have determinant 1. Again, this is a matrix Lie group.

Example 1.1.5

Let $\mathfrak{t}(n,F)$ be the upper triangular matrices in $\mathfrak{gl}(n,F)$ (A matrix A is said to be upper triangular if $x_{ij} = 0$ whenever i > j .). This is Lie algebra with same Lie bracket as $\mathfrak{gl}(n,F)$.

Example 1.1.6

(The unitary and special unitary groups, $U(n)$ and $SU(n)$) An $n \times n$ complex matrix A is unitary if the column vectors of A are orthogonal that is, if

$$\sum_{i=1}^{n} \overline{A_{ij}} A_{ik} = \delta_{jk}$$

Similar to an orthogonal matrix, a unitary matrix has two another equivalent definitions. A matrix A is unitary,

1) If it preserves an inner product.
2) If $A^*A = 1$, i.e. if $A^* = A^{-1}$ (where A^* is adjoint of A)

Since $\det A^* = \overline{\det A}$, $|\det A| = 1$ for all unitary matrices A. This shows unitary matrices are invertible. The same argument as for the orthogonal group can used to show that the set of unitary matrices form a group, called unitary group $U(n)$. This clearly a subgroup of $GL(n,\mathbb{C})$ and since limit of unitary matrices is unitary, $U(n)$ is matrix Lie group. The subgroup of unitary group whose matrices have determinant 1 is the **special unitary group** $SU(n)$. It is easy to see that this is also a matrix Lie group.

The **special unitary Lie algebra** $\mathfrak{su}(n)$ consists of $n\times n$ skew-Hermitian matrices with trace zero. This (real) Lie algebra has dimension n^2-1.

1.2. Lie algebras of derivations

By an F-algebra we simply mean a vector space U over F endowed with a bilinear operation $U\times U\rightarrow U$. **Derivation** of U is a linear mapping δ : $U\rightarrow U$ satisfying the product rule , δ (ab) = a δ(b) + δ(a) b , for all a,b ϵ U.
The collection Der$U$ of all derivations of U is a vector space of End U.

The commutator [δ , δ'] of two derivations is again a derivation . So Der U is a subalgebra of $\mathfrak{gl}(U)$.

If $x \in L$, $y \mapsto [x, y]$ is an endomorphism of L denoted by ad x . By Jacobi identity we have,

(ad x) [y, z] = [x, [y, z]]

= [[x, y] , z] + [y,[x, z]]

= [(ad x) y, z] + [y ,(ad x) z] , for all y,z ϵ L

Therefore ad x is a derivation of L.

In fact ad x ϵ Der L. Derivations of this form are called inner and all others outer.

The map $L \to$ Der L sending x to ad x is called the **adjoint representation of L**.

1.3. Abstract Lie algebra

If L is a algebra over a field F with basis ($x_1, x_2, \dots, x_n$) then [,] is completely determined by the products $[x_i, x_j]$ .We define scalars $a_{ij}^k \in \mathrm{F}$ such that , $[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k$.

The a_{ij}^k are the structure constants of L with respect to the basis . We emphasise that the a_{ij}^k depend on the choice of basis of L. Different bases will in general give different structure constants .

By L_2 and L_2' , $[x_i, x_i] = 0$, for all i and

$[x_i, x_j] = - [x_j , x_i]$, for all i,j .

So it is sufficient to know the structure constants a_{ij}^k for $1 \leq i < j \leq n$.

Thus , it is possible to define an abstract Lie algebra from scratch simply by specifying a set of structure constants .

1.4. Ideals and Homomorphisms

1.4.1. Ideals

A subspace I of a Lie algebra L is called an ideal of L if $x \in L, y \in I$ together imply $[x, y] \in I$.

Obviously , O and L itself are ideals of L. A less trivial example is center

$Z(L) = \{ z \in L / [x, z] = 0$ for all $x \in L\}$.

L is abelian iff $Z(L) = L$.

Another important example is the derived algebra of L, denoted $[L, L]$, which is analogous to the commutator subgroup of a group . It consists of all linear combinations of commutators $[x, y]$ and is clearly an ideal .

If I, J are two ideals of a Lie algebra L, then $I + J = \{x+y / x \in I, y \in J\}$ is also an ideal. Similarly, $[I, J] = \{ \sum x_i y_i / x_i \in I, y_i \in J\}$ is an ideal .

If L has no ideals except itself and 0, and if more over $[L, L] \neq 0$, we call L simple . Clearly, L simple implies $Z(L) = 0$ and $L = [L, L]$.

Call $x \in \text{End}(V)$ semisimple if the roots of its minimal polynomial over F are all distinct.

If I is an ideal of the Lie algebra L, then I is in particular a subspace of L and so we may consider the cosets $z + I = \{ z+x \,;\, x \in I\}$ for $z \in L$ and the quotient vector space $L/I = \{ z + I ; z \in L\}$. We claim that a Lie bracket on L/I may be defined by,

$[w + I, z + I] = [w, z] + I$, for $w, z \in L$.

Here the bracket on the RHS is the Lie bracket in L.

Suppose $w + I = w' + I$ and $z + I = z' + I$. Then , $w - w' \in I$ and $z - z' \in I$.

By bilinearity of the Lie bracket in L,

$[w', z'] = [w' + (w-w'), z' + (z-z')]$

$= [w, z] + [w-w', z'] + [w', z-z'] + [w-w', z-z']$

where the final 3 summands all belongs to I. Therefore $[w'+I, z'+I] = [w, z] + I$, which implies that $[w, z] + I$ depends only on the cosets containing w and z and not on the particular coset representatives w and z. Thus , Lie bracket on L/I is a Lie algebra called the quotient or factor algebra of L by I.

The normalizer of a subalgebra K of L is defined by ,

$N_L(K) = \{ x \in L / [x, K] \subset K\}$.

By the Jacobi identity , $N_L(K)$ is a subalgebra of L and it is the largest subalgebra of L which includes K as an ideal.

If $K = N_L(K)$, we call K self – normalizing. The centralizer of a subset X of L is $C_L(X) = \{x \in L / [x, x] = 0\}$. Again by the Jacobi identity , $C_L(X)$ is a subalgebra of L.

1.4.2. Homomorphisms and Representations

A linear transformation $\phi: L \to L'$ is called a **homomorphism** if $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x,y \in L$. ϕ is called a **monomorphism** if Kernal(ϕ) = 0 , an **epimorphism** if Image (ϕ) = L' , an **isomorphism** if it is both monomorphism and epimorphism.

A **representation** of a Lie algebra L is a homomorphism $\phi: L \to \mathfrak{gl}(V)$ where V is a vector space over F.

The adjoint representation ad:$L \to \mathfrak{gl}(L)$ is an example of representation of a Lie algebra .Clearly , ad is a linear transformation.

Consider, $[\text{ad}x, \text{ad}\, y](z) = \text{ad}\, x\, \text{ad}\, y\,(z) - \text{ad}\, y\, \text{ad}\, x\,(z)$

$= \text{ad}\, x\,([y, z]) - \text{ad}\, y\,([x, z])$

$= [x, [y, z]] - [y, [x, z]]$

$= [x, [y, z]] + [[x, z], y]$ (by L2)

$= [[x, y], z]$ (by L3)

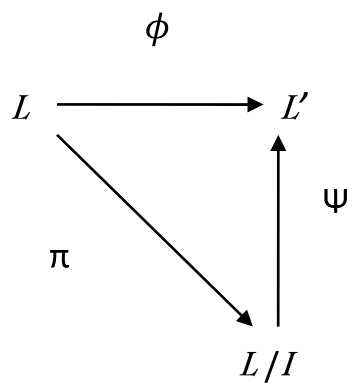
$= \text{ad}\,[x, y]\,(z)$

Thus, ad preserves the bracket.

It consists of all $x \in L$ for which ad x = 0. i.e., for which $[x, y] = 0$ (for all $y \in L$).
So Ker (ad) = $Z(L)$.

Proposition 1.4.2.1

a) If $\phi : L \to L'$ is a homomorphism of Lie algebras , then $L/\ker \phi \cong \operatorname{Im} \phi$.
If I is any ideal of L induced in ker ϕ, there exist a unique homomorphism $\Psi : L/I \to L'$ making the following diagram commute.(π = canonical map):



b) If I and J are ideals of L such that $I \subset J$, then I/J is an ideal of L/I and $(L/I)/(J/I)$ is mutually isomorphic to L/J.

c) If I, J are ideals of L, there is a natural isomorphism between $(I+J)/J$ and $I/(I \cap J)$.

Note 1.4.2.1

Any simple Lie algebra is isomorphic to a linear Lie algebra.

1.4.3. Automorphisms

An **automorphism** of L is an isomorphism of L onto itself . Aut L denotes the group of all such.

If $g \in GL(V)$, the general linear group consisting of all endomorphisms of V, and if $gLg^{-1} = L$ then $x \mapsto gxg^{-1}$ is an automorphism of L.

1.5. Solvable and Nilpotent Lie algebras

1.5.1. Solvability

Lemma 1.5.1.1

Suppose that I is an ideal of L. Then L/I is an ideal iff I contains the derived algebra L' .

Proof

The algebra L/I is abelian iff for all $x,y \in L$ we have ,

$[x + I, y + I] = [x, y] + I = I$.

or ,equivalently, for all $x,y \in L$ we have $[x, y] \in I$.Since I is a subspace of L , this hold iff the space spanned by the brackets $[x, y]$ is contained in I;
i.e., $L' \subseteq I$.

Hence the proof.

Definition 1.5.1.1

L' is the smallest ideal of L with an abelian quotient . The derived algebra L' itself has a smallest ideals whose quotient is abelian , namely the derived algebra of L' ,which we denote $L^{(2)}$ and so on. The derived series of L is defined as, $L^{(1)} = L'$ and $L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$ for $k \geq 2$ .Then $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq ...$

As the product of ideals is an ideal , $L^{(k)}$ is an ideal of L.

The Lie algebra L is said to be **solvable** if $L^{(n)} = 0$ for some n.

Abelian implies solvable, whereas simple algebras are definitely non solvable. The algebra $\mathfrak{t}(n, \mathrm{F})$ of upper triangular matrices is solvable.

Proposition 1.5.1.2

Let L be a Lie algebra.

a) If L is solvable , then so are all sub algebras and homomorphic images of L.
b) If I is a solvable ideal of L such that L/I is solvable, then L itself is solvable.
c) If I, J are solvable ideals of L, then so is $I+J$.

Proof

a) From the definition , if K is a sub algebra of L, then $K^{(i)} \subset L^{(i)}$. Similarly, if $\phi : L \rightarrow M$ is an epimorphism, an easy induction on i shows that $\phi(L^{(i)}) = M^{(i)}$.
b) Say $(L/I)^{(n)} = 0$. Apply part (a) to the canonical homomorphism $\pi: L \rightarrow L/I$,we get $\pi(L^{(n)}) = 0$ or $L^{(n)} \subset I = \mathrm{Ker}\,\pi$. Now if $I^{(m)} = 0$, the obvious fact that $(L^{(i)})^{(j)} = L^{(i+j)}$ implies that , $L^{(n+m)} = 0$ (apply proof of part (a) to the situation $L^{(n)} \subset I$).
c) One of the standard homomorphism theorems yields an isomorphism between $(I+J)/J$ and $I/(I \cap J)$. As a homomorphic image of I, the right side is solvable, so $(I+J)/J$ is solvable. Then so is $I+J$, by part (b) applied to the pair $I+J$, J.

Hence the proof

Definition 1.5.1.2

The maximal solvable ideal is said to be the **radical** of L and is denoted Rad L. A non zero Lie algebra L is said to be semisimple if it has no non-zero solvable ideals or equivalently if Rad $L = 0$.

For example , a simple algebra is semisimple . L has no ideals except itself and 0 and L is non sovable .Also , $L = 0$ is semisimple.

1.5.2. Nilpotency

Definition 1.5.2.1

Define the lower central series of a Lie algebra L to be the series with terms $L^1 = L'$ and $L^k = [L, L^{k-1}]$ for $k \geq 2$.
Then $L \supseteq L^1 \supseteq L^2 \supseteq \ldots$. As the product of ideals is an ideal, L^k is even ideal of L (and not just an ideal of L^{k-1}). The reason for the name central series comes from the fact that L^k / L^{k+1} is contained in the centre of L/L^{k+1}.

The Lie algebra L is said to be **nilpotent** if for some $m \geq 1$ we have $L^m = 0$.
For example , any abelian algebra is nilpotent.
Clearly, $L^{(i)} \subset L^i$ for all i , so nilpotent algebra are solvable.

Proposition 1.5.2.1

Let L be a Lie algebra.

(a) If L is nilpotent, then so are all sub algebras and homomorphic images of L.

(b) If $L/Z(L)$ is nilpotent , then so is L.

(c) If L is nilpotent and non zero , then $Z(L) \neq 0$.

Proof

(a) From the definition , if K is a sub algebra of L, then for each i it is clear that $K^1 \subseteq L^1$, so if $L^n = 0$ then also $K^n = 0$. Similarly, if ϕ : $L \to M$ is an epimorphism , an easy induction on i shows that ϕ (L^i) = M^i

(b) Say $L^n \subset Z(L)$, then $L^{n+1} = [L, L^n] \subset [L, Z(L)] = 0$

(c) The last nonzero term of the descending central series is central.

Note 1.5.2.1

The condition for L to be nilpotent can be rephrased as follows:

For some n (depending only on L), ad x_1,ad x_2,...,ad x_n (y) = 0 for all x_i, y ∈ L

In particular, (ad $x)^n = 0$ for all $x \in L$.

Let L is any Lie algebra and $x \in L$ then x is said to be **ad-nilpotent** if adx is a nilpotent endomorphism

If L is nilpotent , then all elements of L are ad-nilpotent.

Chapter 2

Semi simple Lie algebras

2.1. Killing form

2.1.1. Criterion for semisimplicity

Let L be any Lie algebra . if $x,y \in L$ define $\kappa(x, y) = Tr(\text{ad } x \text{ ad } y)$.Then κ is a symmetric bilinear form on L, called the **Killing form** . κ is also **associative** , in the sense that $\kappa([x, y], z) = \kappa(x, [y, z])$.

$Tr([x, y], z) = Tr(x, [y, z])$ for endomorphisms x , y , z of finite dimensional vector space.

Lemma 2.1.1.1

Let I be an ideal of L . If κ is the Killing form of L and κ_I the killing form of I (viewed as Lie algebra), then $\kappa_I = \kappa|_{I \times I}$.

Proof

First , a simple fact from linear algebra : If W is a subspace of a finite dimensional vector space V, and ϕ an endomorphism of V mapping V into W, then $Tr\ \phi = Tr\ (\phi|_W)$.Now if $x,y \in I$, then (ad x)(ad y) is an endomorphism of L ,mapping L into I, so its trace $\kappa(x, y)$ coincides with the trace $\kappa_I(x, y)$ of (ad x)(ad y)$|_I$ = ($\text{ad}_I\ x$)($\text{ad}_I\ y$).
Hence the proof.

Definition 2.1.1.1

A symmetric bilinear form β(x, y) is called **non degenerate** if its **radical** S is 0 where S = {$x \in L$ / β(x, y) = 0 ∀ $y \in L$}.Because the Killing form is associative , its radical is more than just a subspace : S is an *ideal* of L.

Fix a basis $x_1,\ldots,x_n$ of L. Then κ is non degenerate iff the $n \times n$ matrix whose i,j entry is $\kappa(x_i, x_j)$ has nonzero determinant.

Cartan's Criterion

"Let L be a sub algebra of $\mathfrak{gl}(V)$, V finite dimensional. Suppose that $Tr(xy) = 0$ ∀ $x \in [L, L]$, $y \in L$. Then L is solvable ".

Theorem 2.1.1.2

Let L be a Lie algebra . Then L is semisimple iff its Killing form is non degenerate.

Proof

Suppose first that Rad $L = 0$. Let S be the radical of κ. By definition, Tr(ad x ad y) = 0 ∀ $x \in S$, $y \in L$. According to *Cartan's criterion*, $\mathrm{ad}_L S$ is solvable, hence S is solvable. Since S is an ideal of L, so $S \subset$ Rad $L = 0$,and κ is non degenerate.

Conversely, let $S = 0$. To prove that L is semisimple, it will suffice to prove that every abelian ideal I of L is induced in S. Suppose $x \in I$, $y \in L$. Then ad x ad y maps $L \to L \to I$, and (ad x ad y)^2 maps L into $[I, I] = 0$. This means that ad x ad y is nilpotent, hence that

0 = Tr(ad x ad y) = $\kappa(x, y)$, so $I \subset S = 0$.

2.1.2 Simple ideals of L

Definition 2.1.2.1

A Lie algebra L is said to be the **direct sum** of ideals $I_1,\ldots,I_t$ provided $L = I_1+\ldots+I_t$ (direct sum of subspaces). This condition forces $[I_i, I_j] \subset I_i \cap I_j = 0$ if $i \neq j$. We write $L = I_1 \oplus \ldots \oplus I_t$.

Theorem 2.1.2.1

Let L be semisimple. Then there exist ideals $L_1,\ldots,L_t$ of L which are simple (as Lie algebras), such that $L = L_1 \oplus \ldots \oplus L_t$. Every simple ideals of L coincides with one of the L_i. Moreover, the Killing form of L_i is the restriction of κ to $L_i \times L_i$.

Proof

As a first step, let I be an arbitrary ideal of L. Then

$I^{\perp} = \{x \in L \mid \kappa(x, y) = 0 \ \forall \ y \in I\}$ is also an ideal, by the associativity of κ. *Cartan's Criterion*, applied to the Lie algebra I, shows that the ideal $I \cap I^{\perp}$ of L is solvable (hence 0). Therefore, since dim I+dim $I^{\perp}$= dim L, we must have $L = I \oplus I^{\perp}$.

Now proceed by induction on dim L to obtain the desired decomposition into direct sum of simple ideals . If L has no nonzero proper ideals, then L is simple already and we are done. Otherwise let L_1 be a minimal nonzero ideal ; by the preceding paragraph, $L = L_1 \oplus L_1^{\perp}$. In particular, any ideal of L_1 is also an ideal of L, so L_1 is semisimple. For the same reason, $L_1^{\perp}$ is semisimple; by induction, it splits into a direct sum of simple ideals, which are also ideals of L. The decomposition of L follows.

Next we have to prove that these simple ideals are unique . If I is any simple ideal of L, then $[I, L]$ is also an ideal of I, nonzero because $Z(L) = 0$; this forces $[I, L] = I$. On the other hand, $[I, L] = [I, L_1] \oplus ... \oplus [I, L_t]$, so all but one summand must be 0 . Say $[I, L_i] = I$. Then $I \subset L_i$,and $I = L_i$(because L_i is simple). The last assertion of the theorem follows from lemma 2.1.1.1 .
Hence the proof.

2.1.3. Inner derivations

Observation : (*) $[\delta, \text{ad } x] = \text{ad } (\delta x)$, $x \in L$, $\delta \in \text{Der } L$.

Theorem 2.1.3.1

If L is semisimple , then ad L = Der L (i.e., every derivation of L is inner).

Proof

Since L is semisimple, $Z(L) = 0$. Therefore, $L \to \text{ad } L$ is an isomorphism of Lie algebras. In particular, $M = \text{ad } L$ itself has non degenerate Killing form. If $D = \text{Der } L$, we just remarked that $[D, M] \subset M$. This implies that κ_M is the restriction to $M \times M$ of the Killing form κ_D of D. In particular, if

$I = M^{\perp}$ is the subspace of D orthogonal to M under κ_D, then the non degeneracy of κ_M forces $I \cap M = 0$. Both I and M are ideals of D, so we obtain $[I, M] = 0$ .If $\delta \in I$, this forces $\text{ad } (\delta x) = 0$ $\forall x \in L$ (by (*)), so in turn $\delta x = 0$ ($x \in L$) because ad is one to one and $\delta = 0$.
Conclusion: $I = 0$, Der $L = M = \text{ad } L$.
Hence the proof.

2.1.4. Abstract Jordan decomposition

Proposition 2.1.4.1

Let V be a finite dimensional vector space over F, $x \in$ End V.

a) There exist unique x_s ,$x_n \in$ End V satisfying the conditions: $x = x_s + x_n$, x_s is semisimple, x_n is nipotent ,x_s and x_n commute.
b) There exist polynomials p(T) , q(T) in one indeterminate , without constant term , x_s = p(x) , x_n = q(x). In particular, x_s and x_n commute with any endomorphism commuting with x.
c) If A ⊂ B ⊂ V are subspaces and x maps B into A , then x_s and x_n also map B into A.

The decomposition $x = x_s + x_n$ is called the ***Jordan – Chevalley decomposition*** of x or just the ***Jordan decomposition***; x_s,x_n are called(respectively) the semisimple part and nilpotent part of x.

Lemma 2.1.4.2

Let U be a finite dimensional *F*-algebra. Then Der U contains the semisimple and nilpotent parts(in End U) of all its elements.

Abstract Jordan decomposition

In particular, since Der L coincides with ad L while $L \to$ ad L is one to one each $x \in L$ determines unique elements s,$n \in L$ such that ad x = ad s + ad n is the usual Jordan decomposition of ad x (in End L). This means that x = s + n, with [s, n] = 0, s ad-semisimple(i.e., ad S semisimple), n ad-nilpotent.
We write s = x_s , $n = x_n$, and call these the semisimple and nilpotent parts of x.

The abstract decomposition of x just obtained does in fact agree with the usual Jordan decomposition in all such cases.

2.2. Complete reducibility of reprsentations

2.2.1. Modules

A vector space V, endowed with an operation $L \times V \rightarrow V$ (denoted $(x, v) \mapsto x.v$ or just xv) is called an **L-module** if the following conditions are satisfied.

M_1: $(ax + by).v = a(x.v) + b(y.v)$

M_2: $x.(av + bw) = a(x.v) + b(x.w)$

M_3: $[x, y].v = x.y.v - y.x.v$

$$(x,y \in L ;\ v,w \in V;\ a,b \in F)$$

For example, if $\phi: L \rightarrow (V)$ is a representation of L, then V may be viewed as an L-module via the action $x.v = \phi(x)(v)$.

Conversely, given an L-module V, this equation defines a representation $\phi: L \rightarrow (V)$.

A **homomorphism of L- modules** is a linear map $\phi: V \rightarrow W$ such that $\phi(x.v) = x.\phi(v)$. The kernel of such a homomorphism is then an L-submodule of V. When ϕ is an isomorphism of vector spaces, we call it an **isomorphism** of L-modules; in this case the two modules are said to afford **equivalent** representation of L. An L-module V is called **irreducible** if it has precisely two L-submodules.

V is called **completely reducible** if V is a direct sum of irreducible L-submodules.

Lemma 2.2.1.1 (Schur's Lemma)

Let $\phi: L \rightarrow \mathfrak{gl}(V)$ be irreducible. Then the only endomorphisms of V commuting with all $\phi(x)$ ($x \in L$) are the scalars.

2.2.2. Casimir element of a representation

Let L be semisimple and let $\phi: L \to \mathfrak{gl}(V)$ be faithful (i.e., one to one) representation of L. Define a symmetric bilinear form $\beta(x, y) = Tr(\phi(x)\ \phi(y))$ on L. The form β is associative, so in particular its radical S is an ideal of L. Moreover β is non degenerate and by *Cartan's Criterion* we have $\phi(S) \cong S$ is solvable. So, $S = 0$.

Now let L be semisimple , β any non degenerate symmetric associative bilinear form on L. If $(x_1, x_2, \dots, x_n)$ is a basis of L , there is a uniquely determined dual basis $(y_1, y_2, \dots, y_n)$ relative to β, satisfying $\beta(x_i, y_j) = \delta_{ij}$. If $x \in L$, we can write $[x, x_i] = \sum_j a_{ij} x_j$ and $[x, y_i] = \sum_j b_{ij} y_j$.

Using the associativity of β, we compute

$$
\begin{aligned}
a_{ik} = \sum_j a_{ij} \beta(x_j, y_k) &= \beta([x, x_i], y_k) \\
&= \beta(-[x_i, x], y_k) = \beta(x_i, -[x, y_k]) \\
&= -\sum_j b_{kj} \beta(x_i, y_j) \\
&= -b_{ki}
\end{aligned}
$$

If $\phi: L \to \mathfrak{gl}(V)$ is any representation of L, write $c_\phi(\beta) = \sum_i \phi(x_i)\phi(y_i) \in \text{End } V$.

Using the identity (in End V), $[x, yz] = [x, y]z + y[x, z]$ and the fact that $a_{ik} = -b_{ki}$, we obtain:

$$
\begin{aligned}
[\phi(x), c_\phi(\beta)] &= \sum_i [\phi(x), \phi(x_i)]\, \phi(y_i) + \sum_i \phi(x_i)\, [\phi(x), \phi(y_i)] \\
&= \sum_{i,j} a_{ij} \phi(x_j)\, \phi(y_i) + \sum_{i,j} b_{ij}\, \phi(x_i)\, \phi(y_j) = 0 .
\end{aligned}
$$

i.e., $c_\phi(\beta)$ is an endomorphism of V commuting with $\phi(L)$. We can conclude that, Let ϕ: $L \rightarrow \mathfrak{gl}(V)$ be a faithful representation with trace form $\beta(x, y) = Tr(\phi(x)\,\phi(y))$. In this case, having fixed a basis $(x_1, x_2, \dots, x_n)$ of L, we write simply c_ϕ for $c_\phi(\beta)$ and call this the **Casimir element of ϕ**. Its trace is

$$\sum_i Tr(\phi(x_i)\,\phi(y_i)) = \sum_i \beta(x_i, y_i) = \dim L.$$

Chapter 3

The Root Space Decomposition

3.1. Preliminary Results

Suppose that L is a complex semisimple Lie algebra containing an abelian subalgebra H consisting of semisimple elements.

We have seen that L has a basis of common eigenvectors for the elements of ad H. Given a common eigen vector $x \in L$, the eigen values are given by the associated weight, $\alpha : H \to L$, defined by

(ad h)x = α(h)x for all h $\in H$.

Weights are elements of the dual space H^*. For each $\alpha \in H^*$, let

$L_\alpha := \{x \in L : [h, x] = \alpha(h)x$ for all $h \in H\}$

denote the corresponding weight space. One of these weight spaces is the zero weight space.

$L_0 = \{ z \in L : [h, z] = 0$ for all $h \in H\}$.

This is the same as the centraliser of H in L, $C_L(H)$. As H is abelian, we have $H \subseteq L_0$.

Let ϕ denote the set of non -zero $\alpha \in L^*$ for which L_α is non zero. We can write the decomposition of L into weight spaces for H as,

$$L = L_0 \oplus \bigoplus_{\alpha \in \phi} L_\alpha \quad \text{..................}(*)$$

Since L is finite-dimensional, this implies that ϕ is finite.

Lemma 3.1.1

Suppose that $\alpha, \beta \in H^*$. Then

(i) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$

(ii) If $\alpha + \beta \neq 0$, then $\kappa(L_\alpha, L_\beta) = 0$

(iii) The restriction of κ to L_0 is non-degenerate.

Proof

(i) Take $x \in L_\alpha$ and $y \in L_\beta$. We must show that $[x, y]$, if non-zero, is an eigenvector for each ad h $\in H$, with eigenvalue $\alpha(h) + \beta(h)$. Using the Jacobi identity we get

$$
\begin{aligned}
[h, [x, y]] &= [[h, x], y] + [x, [h, y]] \\
&= [\alpha(h)x, y] + [x, \beta(h)y] \\
&= \alpha(h)[x, y] + \beta(h)[x, y] \\
&= (\alpha + \beta)(h)[x, y]
\end{aligned}
$$

(ii) Since $\alpha + \beta \neq 0$, there is some h $\in H$ such that $(\alpha + \beta)(h) \neq 0$. Now, for any $x \in L_\alpha$ and $y \in L_\beta$, we have, using the associativity of the Killing form,

$\alpha(h)\,\kappa(x, y) = \kappa([h, x], y) = -\kappa([x, h], y) = -\kappa(x, [h, y]) = -\beta(h)\kappa(x, y)$, and hence

$$(\alpha + \beta)(h)\kappa(x, y) = 0$$

Since by assumption $(\alpha + \beta)(h) \neq 0$, we must have $\kappa(x, y) = 0$.

(iii) Suppose that $z \in L_0$ and $\kappa(z, x) = 0$ for all x in L_0. By (ii) we know that $L_0 \perp L_\alpha$ for all $\alpha \neq 0$. If $x \in L$, then by (*) we can write x as

$$x = x_0 + \sum_{\alpha\in\phi} x_\alpha$$

with $x_\alpha \in L_\alpha$. By linearity, $\kappa(z, x) = 0$ for all $x \in L$. Since κ is non-degenerate, it follows that $z = 0$, as required.

Hence the proof.

Definition 3.1.1

A Lie sub algebra H of a Lie algebra L is said to be a ***Cartan subalgebra*** (or **CSA**) if H is abelian and every element h ϵ H is semisimple, and moreover H is maximal with these properties.

Note 3.1.1

We do not assume L is semisimple in this definition.

Remark 3.1.1

Cartan subalgebra is same as the *maximal toral subalgebra* , where *toral subalgebra* is defined as follows. A semisimple Lie algebra L possess nonzero subalgebras consisting of semisimple elements then such a subalgebra is called ***toral***. A *maximal toral subalgebra* H of L is a *toral subalgebra* not properly included in any other.

3.2. Cartan Subalgebras

Lemma 3.2.1

Let H be a *Cartan subalgebra* of L. Suppose that h ϵ H is such that the dimension of C_L(h) is minimal. Then every s ϵ H is central in C_L(h), and so C_L(h) ⊆ C_L(s). Hence, C_L(h) = $C_L(H)$.

Proof

We shall show that if s is not central in C_L(h), then there is a linear combination of s and h whose centraliser has smaller dimension than C_L(h).

First we construct a suitable basis for L. We start by taking a basis of C_L(h) ∩ C_L(s), {c_1,...,c_n} , say. As s is semisimple and s ϵC_L(h), ad s acts diagonalisably on C_L(h). We may therefore extend this basis to a basis of C_L(h) consisting of ad s eigenvectors, say by adjoining {x_1,...,x_p}. Similarly we may extend $\{c_1,...,c_n\}$ to a basis of C_L (s) consisting of ad h eigenvectors, say by adjoining $\{y_1,...,y_q\}$. We can prove that $\{c_1,...,c_n,x_1,...,x_p,y_1,...,y_q\}$ is a basis of C_L (h) + C_L(s). Finally, as ad h and ad s commute and act diagonalisably on L, we may extend this basis to a basis of L by adjoining simultaneous eigenvectors for ad h and ad s, say $\{w_1,...,w_r\}$.

Note that if [s, x_j] = 0 then $x_j \in C_L$(s) $\cap C_L$(h) ,a contradiction. Similarly, we can see that [h, y_k] $\neq$ 0. Let [h, w_l] = $\theta_l w_l$ and [s, w_l] = $\sigma_l w_l$. Again we have $\theta_l, \sigma_l \neq 0$ for $1 \leq l \leq r$.

The following table summarises the eigen values of ad s, ad h and
ad s + λ ad h, where λ ≠ 0 :

	c_i	x_j	y_k	w_l
ad s	0	$\neq 0$	0	σ_l
ad h	0	0	$\neq 0$	θ_l
ad s + λ ad h	0	$\neq 0$	$\neq 0$	$\sigma_l + \lambda\,\theta_l$

Thus, if we choose λ so that λ ≠ 0 and $\lambda \neq -\sigma_l/\theta_l$ for any l, then we will have

$$C_L(s + \lambda h) = C_L(s) \cap C_L(h).$$

By hypothesis, C_L(h) $\nsubseteq$ C_L(s), so this subspace is of smaller dimension than C_L(h); this contradicts the choice of h.

Now, since $C_L(H)$ is the intersection of the C_L(s) for s ∈ H, it follows that C_L(h) ⊆ $C_L(H)$. The other inclusion is obvious, so we have proved that C_L(h) = $C_L(H)$.
Hence the proof.

Theorem 3.2.2

If H is a *Cartan subalgebra* of L and h ∈ H is such that C_L(h) = $C_L(H)$, then C_L(h) = H. Hence H is self-centralising.

Proof

Since H is abelian, H is certainly contained in C_L(h). Suppose $x \in C_L$(h) has abstract Jordan decomposition x = s + n. As x commutes with h, by a theorem which state that,

"Let L be a complex semisimple Lie algebra. Each $x \in L$ can be written uniquely as $x = d + n$, where $d, n \in L$ are such that ad d is diagonalisable, ad n is nilpotent, and $[d, n] = 0$. Furthermore, if $y \in L$ commutes with x, then $[d, y] = 0$ and $[n, y] = 0$".

It will implies that both s and n lie in C_L(h),so we must show that s ∈ H and n = 0.

We almost know already that s ∈H. Namely, Since C_L(h) = $C_L(H)$, we have that s commutes with every element of H and therefore H+ Span{s} is an abelian sub algebra of Lconsisting of semisimple elements. It contains the *Cartan subalgebra* H and hence by maximality s ∈ H.

To show that the only nilpotent element in $C_L(H)$ is 0 takes slightly more work.

Step 1 : C_L(h) *is nilpotent*. Take $x \in C_L$(h) with x = s + n as above. Since s ∈ H, it must be central in C_L(h), so, regarded as linear maps from C_L(h) to itself, we have ad x = ad n. Thus for every $x \in C_L$(h), ad x : C_L(h) → C_L(h) is nilpotent. It now follows from the *second version of Engel's theorem* which states that

"A Lie algebra L is nilpotent iff for all $x \in L$ the linear map ad x : $L \to L$ is nilpotent ".

It implies that C_L(h) is a nilpotent Lie algebra.

Step 2 : *Every element in $C_L(h)$ is semisimple*. Let $x \in C_L$(h) have abstract Jordan decomposition x = s + n as above. As C_L(h) is nilpotent, it is certainly solvable, so by *Lie's theorem* which states that,

"Let V be an n-dimensional complex vector space and let L be a solvable Lie sub algebra of $\mathfrak{gl}(V)$ .Then there is a basis of V in which every element of L is represented by an upper triangular matrix " we can see that there is a basis of L in which the maps ad x for $x \in C_L$(h) are represented by upper triangular matrices. As ad n: $L \to L$ is nilpotent, its matrix must be strictly upper triangular. Therefore

$$\kappa(n, z) = Tr(\text{ad}\, n \circ \text{ad}\, z) = 0$$

for all $z \in C_L$(h). By Lemma 3.1.1 (iii), the restriction of κ to $C_L(H)$ is non-degenerate, so we deduce *n* = 0, as required.
Hence the proof.

3.3. Definition of the Root Space Decomposition

Let H be a *Cartan sub algebra* of our semisimple Lie algebra L. As H= $C_L(H)$, the direct sum decomposition of L into weight spaces for H may be written as

$$L = H \oplus \bigoplus_{\alpha \in \phi} L_\alpha ,$$

where ϕ is the set of $\alpha \in H^*$ such that $\alpha \neq 0$ and $L_\alpha \neq 0$. Since L is finite-dimensional, ϕ is finite.

If $\alpha \in \phi$, then we say that α is a *root* of L and L_α is the associated *root space*. The direct sum decomposition above is the *root space decomposition*. It should be noted that the roots and root spaces depend on the choice of *Cartan subalgebra* H.

3.4. Subalgebras Isomorphic to $\mathfrak{sl}(2, \mathbb{C})$

We shall now associate to each root $\alpha \in \phi$ a Lie subalgebra of L isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. These sub algebras will enable us to deduce several results on the structure of L.

Note 3.4.1

Suppose that L is a complex Lie algebra of dimension 3. Then , $\mathfrak{sl}(2,\mathbb{C})$ is the only one 3-dimensional complex Lie algebra with $L' = L$.

Lemma 3.4.1

Suppose that $\alpha \in \phi$ and that x is a non-zero element in L_α. Then $-\alpha$ is a root and there exists $y \in L_{-\alpha}$ such that Span$\{x, y, [x, y]\}$ is a Lie subalgebra of L isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

Proof

First we claim that there is some $y \in L_{-\alpha}$ such that $\kappa(x, y) \neq 0$ and $[x, y] \neq 0$. Since κ is non-degenerate, there is some $w \in L$ such that $\kappa(x,w) \neq 0$. Write $w = y_0 + \sum_{\beta\in\phi} y_\beta$ with $y_0 \in L_0$ and $y_\beta \in L_\beta$. When we expand $\kappa(x, y)$, we find by Lemma 3.1.1(ii) that the only way a non-zero term can occur is if $-\alpha$ is a root and $y_{-\alpha} \neq 0$, so we may take $y = y_{-\alpha}$. Now, since α is non-zero, there is some $t \in H$ such that $\alpha(t) \neq 0$. For this t, we have

$$\kappa(t, [x, y]) = \kappa([t, x], y) = \alpha(t)\, \kappa(x, y) \neq 0$$

and so $[x, y] \neq 0$.

Let $S :=$ Span$\{x, y, [x, y]\}$. By Lemma 3.1.1(i), $[x, y]$ lies in $L_0 = H$. As x and y are simultaneous eigenvectors for all elements of ad H, and so in particular for ad $[x, y]$, this shows that S is a Lie sub algebra of L. It remains to show that S is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

Let $h := [x, y] \in S$. We claim that $\alpha(h) \neq 0$. If not, then $[h, x] = \alpha(h)x = 0$; Similarly $[h, y] = -\alpha(h)x = 0$, so ad $h : L \to L$ commutes with ad $x : L \to L$ and ad $y : L \to L$. By a Proposition which state that

“ Let x , y : $V \to V$ be linear maps from a complex vector space V to itself . Suppose that x and y both commute with $[x, y]$. Then $[x, y]$ is a nilpotent map .”
we get, ad h : $L \to L$ is a nilpotent map.

On the other hand, because H is a *Cartan subalgebra*, h is semisimple. The only element of L that is both semisimple and nilpotent is 0, so $h = 0$, a contradiction.

Thus S is a 3-dimensional complex Lie algebra with $S' = S$. By the note 3.4.1, S is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.
Hence the Lemma.

3.5. Root Strings and Eigenvalues

Given $h \in H$, let θ_h denote the map $\theta_h \in H^*$ defined by

$$\theta_h(k) = \kappa(h, k) \quad \text{for all } k \in H.$$

By Lemma 3.1.1(iii) the Killing form is non-degenerate on restriction to H, so the map $h \mapsto \theta_h$ is an isomorphism between H and H^*.
In particular, associated to each root $\alpha \in \phi$ there is a unique element $t_\alpha \in H$ such that

$$\kappa(t_\alpha, k) = \alpha(k) \quad \text{for all } k \in H.$$

Lemma 3.5.1

Let $\alpha \in \phi$. If $x \in L_\alpha$ and $y \in L_{-\alpha}$, then $[x, y] = \kappa(x, y)t_\alpha$. In particular, $h_\alpha = [e_\alpha, f_\alpha] \in \text{Span}\{t_\alpha\}$.

Proof

For $h \in H$, we have

$$\kappa(h, [x,y]) = \kappa([h, x],y) = \alpha(h)\kappa(x , y) = \kappa(t_\alpha, h)\kappa(x, y).$$

Now we view $\kappa(x, y)$ as a scalar and rewrite the right-hand side to get

$$\kappa(h, [x, y]) = \kappa(h, \kappa(x, y)t_\alpha).$$

This shows that $[x, y] - \kappa(x, y)t_\alpha$ is perpendicular to all $h \in H$, and hence it is zero as κ restricted to H is non-degenerate.
Hence the proof.

Chapter 4

Root systems

4.1. Definition of Root Systems

Let E be a finite-dimensional real vector space endowed with an inner product written $(-,-)$. Given a non-zero vector $v \in E$, let s_v be the reflection in the hyperplane normal to v. Thus s_v sends v to $-v$ and fixes all elements y such that $(y, v) = 0$. We can also see that

$$s_v(x) = x - \frac{2(x,v)}{(v,v)} v \quad \text{for all} \quad x \in E$$

and that s_v preserves the inner product, that is,

$$(s_v(x), s_v(y)) = (x, y) \quad \text{for all} \quad x, y \in E.$$

As it is a very useful convention, we shall write

$$< x , v > := \frac{2\,(x,v)}{(v,v)} ,$$

noting that the symbol $<x,v>$ is only linear with respect to its first variable, x.

Definition 4.1.1

A subset R of a real vector space E is a ***root system*** if it satisfies the following axioms.

(R_1) R is finite, it spans E, and it does not contain 0.

(R_2) If $\alpha \in R$, then the only scalar multiples of α in R are $\pm\alpha$.

(R_3) If $\alpha \in R$, then the reflection s_α permutes the elements of R.

(R_4) If $\alpha, \beta \in R$, then $< \beta,\alpha > \in \mathbf{Z}$.

The elements of R are called ***roots***.

4.2. First Steps in the Classification

Lemma 4.2.1 (Finiteness Lemma)

Suppose that R is a root system in the real inner-product space E. Let $\alpha, \beta \in R$ with $\beta \neq \pm\alpha$. Then

$$< \alpha, \beta><\beta,\alpha> \in \{0, 1, 2, 3\}.$$

Proof

The product in question is an integer (by R 4). We must establish the bounds. For any non-zero $v, w \in E$, the angle θ between v and w is such that $(v, w)^2 = (v, v)(w, w) \cos^2 \theta$. This gives

$$< \alpha, \beta><\beta,\alpha> = 4\cos^2 \theta \le 4.$$

Suppose we have $\cos^2 \theta = 1$. Then θ is an integer multiple of π and so α and β are linearly dependent, contrary to our assumption.
Hence the proof.

Note 4.2.1

Take two roots α, β in a root system R with $\alpha \neq \pm\beta$. We may choose the labelling so that $(\beta, \beta) \ge (\alpha, \alpha)$ and hence

$$|<\beta,\alpha>| = \frac{2\,|\,(\beta,\alpha)\,|}{(\alpha,\alpha)} \ge \frac{2\,|\,(\alpha,\beta)\,|}{(\beta,\beta)} = |<\alpha,\beta>|$$

By the *Finiteness Lemma* , the possibilities are:

$<\alpha,\beta>$	$<\beta,\alpha>$	θ	$\frac{(\beta,\beta)}{(\alpha,\alpha)}$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Proposition 4.2.2

Let $\alpha, \beta \in R$.

(a) If the angle between α and β is strictly obtuse, then $\alpha + \beta \in R$.

(b) If the angle between α and β is strictly acute and $(\beta, \beta) \geq (\alpha, \alpha)$, then $(\alpha - \beta) \in R$.

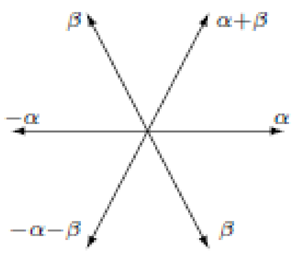
Proof

In either case, we may assume that $(\beta, \beta) \geq (\alpha, \alpha)$. By ($R_3$), we know that $s_\beta(\alpha) = \alpha - \langle\alpha, \beta\rangle\beta$ lies in R. The table shows that if θ is strictly acute, then $\langle\alpha, \beta\rangle = 1$, and if θ is strictly obtuse, then $\langle\alpha, \beta\rangle = -1$.
Hence the proof.

Example 4.2.1

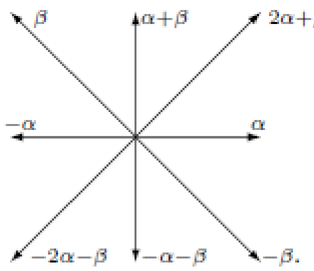
Let $E = \mathbf{R}^2$ with the Euclidean inner product. We shall find all root systems R contained in E. Take a root α of the shortest possible length. Since R spans E, it must contain some root $\beta \neq \pm\alpha$. By considering $-\beta$ if necessary, we may assume that β makes an obtuse angle with α. Moreover, we may assume that this angle, say θ, is as large as possible.

a) Suppose that $\theta = 2\pi/3$. Using Proposition 4.2.2, we find that R contains the six roots shown below.



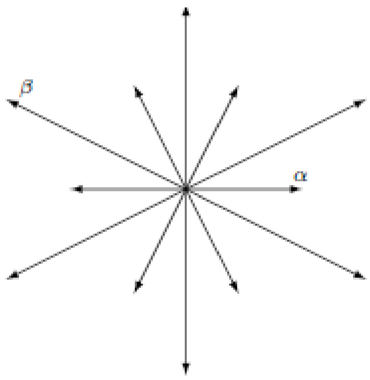
One can check that this set is closed under the action of the reflections $s_\alpha, s_\beta, s_{\alpha+\beta}$. As $s_{-\alpha} = s_\alpha$, and so on, this is sufficient to verify (R_3). We have therefore found a root system in E.

b) Suppose that $\theta = 3\pi/4$. Proposition 4.2.2 shows that $\alpha + \beta$ is a root, and applying s_α to β shows that $2\alpha + \beta$ is a root, so R must contain



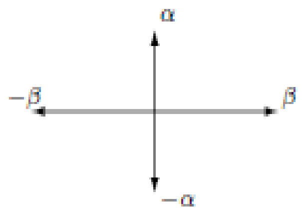
A further root would make an angle of at most $\pi/8$ with one of the existing roots, so this must be all of R.

c) Suppose that $\theta = 5\pi/6$. One can show that R must be



and to determine the correct labels for the remaining roots.

(d) Suppose that β is perpendicular to α.



Here, as $(\alpha, \beta) = 0$, the reflection s_α fixes the roots $\pm\beta$ lying in the space perpendicular to α, so there is no interaction between the roots $\pm\alpha$ and $\pm\beta$. In particular, knowing the length of α tells us nothing about the length of β.

Definition 4.2.1

The root system R is ***irreducible*** if R cannot be expressed as a disjoint union of two non-empty subsets $R_1 \cup R_2$ such that $(\alpha, \beta) = 0$ for $\alpha \in R_1$ and $\beta \in R_2$.

Note 4.2.2

If such a decomposition exists, then R_1 and R_2 are root systems in their respective spans.

Lemma 4.2.3

Let R be a root system in the real vector space E. We may write R as a disjoint union

$$R = R_1 \cup R_2 \cup \ldots \cup R_k ,$$

where each R_i is an irreducible root system in the space E_i spanned by R_i, and E is a direct sum of the orthogonal subspaces $E_1, \ldots, E_k$.

Proof

Define an equivalence relation $\sim$ on R by letting $\alpha \sim \beta$ if there exist

$\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_s$ in R with $\alpha = \Upsilon_1$ and $\beta = \Upsilon_s$ such that $(\Upsilon_i, \Upsilon_{i+1}) \neq 0$ for $1 \leq i < s$.

Let the R_i be the equivalence classes for this relation. It is clear that they satisfy axioms (R_1), (R_2), and (R_4). That each R_i is irreducible follows immediately from the construction.

As every root appears in some E_i, the sum of the E_i spans E. Suppose that $v_1 + \ldots + v_k = 0$, where $v_i \in E_i$. Taking inner products with v_j, we get

$$0 = (v_1, v_j) + \ldots + (v_j, v_j) + \ldots + (v_k, v_j) = (v_j, v_j)$$

so each $v_j = 0$. Hence $E = E_1 \oplus \ldots \oplus E_k$.

4.3 Bases for Root Systems

Let R be a root system in the real inner- product space E. Because R spans E, any maximal linearly independent subset of R is a vector space basis for R. Proposition 4.2.2 suggests that it might be convenient if we could find such a subset where every pair of elements made an obtuse angle .

Definition 4.3.1

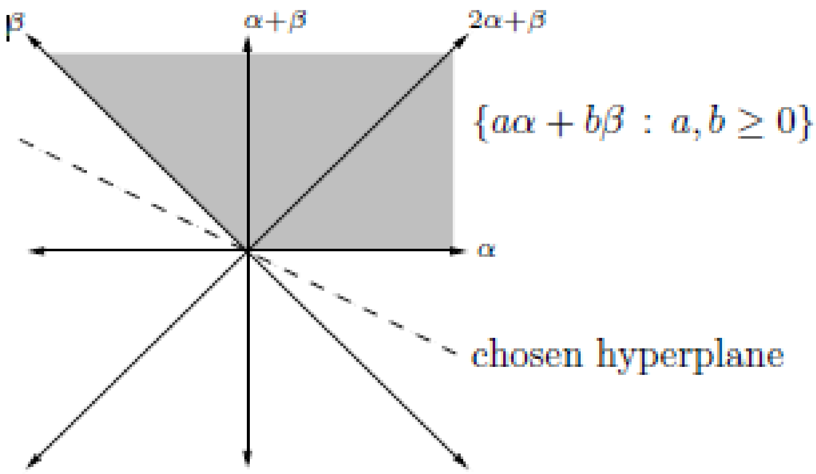
A subset B of R is a *base* for the root system R if

(B1) B is a vector space basis for E, and

(B2) every $\beta \in R$ can be written as $\beta = \sum_{\alpha \in B} k_\alpha \alpha$ with $k_\alpha \in \mathbf{Z}$, where all the non-zero coefficients k_α have the same sign.

Note 4.3.1

A natural way to label the elements of R as positive or negative is to fix a hyperplane of codimension 1 in E which does not contain any element of R and then to label the roots of one side of the hyperplane as positive and those on the other side as negative. Suppose that R has a base B compatible with this labelling. Then the elements of B must lie on the positive side of the hyperplane. For example, the diagram below shows a possible base for the root system in example 4.2.1 (b).



Note 4.3.2

The roots in the base are those nearest to the hyperplane.

Theorem 4.3.1

Every root system has a base.

Proof

Let R be a root system in the real inner-product space E. We may assume that E has dimension at least 2, as the case dim E = 1 is obvious. We may choose a vector $z \in E$ which does not lie in the perpendicular space of any of the roots. Such a vector must exist , as E has dimension at least 2, so it is not the union of finitely many hyperplanes .

Let R^+ be the set of $\alpha \in R$ which lie on the positive side of z, that is, those α for which $(z,\alpha) > 0$. Let

$$B := \{\alpha \in R^+ : \alpha \text{ is not the sum of two elements in } R^+\}.$$

We claim that B is a base for R.

We first show that (B2) holds . If $\beta \in R$, then either $\beta \in R^+$ or $-\beta \in R^+$, so It is sufficient to prove that every $\beta \in R^+$ can be expressed as $\beta = \sum_{\alpha \in B} k_\alpha \alpha$ for some $k_\alpha \in \mathbf{Z}$ with each $k_\alpha \geq 0$. If this fails , then we may choose a $\beta \in R^+$,not expressible in this form, such that the inner product (z, β) is as small as possible. As $\beta \notin B$, there exist $\beta_1, \beta_2 \in R^+$ such that $\beta = \beta_1 + \beta_2$. By linearity,

$$(z, \beta) = (z, \beta_1) + (z, \beta_2)$$

Is the sum of two positive numbers, and therefore $0 < (z, \beta_i) < (z, \beta)$ for $i = 1, 2$. Now at least one of β_1, β_2 cannot be expressed as a positive integral linear combination of the elements of α; this contradicts the choice of β.

It remains to show that B is linearly independent. First note that if α and β are distinct elements of B, then the angle between them must be obtuse. Suppose that $\sum_{\alpha \in B} \gamma_\alpha \alpha = 0$, where $\boldsymbol{\gamma}_\alpha \in \mathbf{R}$. Collecting all the terms with positive coefficients to one side gives an element

$$x := \sum_\alpha \gamma_\alpha \alpha = \sum_\beta (-\gamma_\beta) \beta \;\; ; \;\; \boldsymbol{\gamma}_\alpha > 0 \, , \boldsymbol{\gamma}_\beta < 0$$

Hence

$$(x, x) = \sum_{\alpha, \beta} \gamma_\alpha(-\boldsymbol{\gamma}_\beta)(\alpha, \beta) \leq 0 \;\; ; \;\; \boldsymbol{\gamma}_\alpha > 0 \, , \boldsymbol{\gamma}_\beta < 0$$

and so $x = 0$. Therefore

$$0 = (x, z) = \sum_\alpha \gamma_\alpha (\alpha, z) \;\; ; \boldsymbol{\gamma}_\alpha > 0$$

where each $(\alpha, z) > 0$ as $\alpha \in R^+$, so we must have $\boldsymbol{\gamma}_\alpha = 0$ for all α, and similarly $\boldsymbol{\gamma}_\beta = 0$ for all β.

Chapter 5

Applications

Lie algebra and its representation theory play a vital role in various branches of mathematics such as harmonic analysis, algebraic topology, geometry etc. and also in the theory of particle physics. In addition, it also helped to mathematical discipline of topology. Since Lie algebra has a wide range of applications only few of them are discussed here.

5.1. Harmonic oscillators and current

Lie algebras arise in the study of conservation laws in classical mechanics. The modern theory of conservation laws in classical mechanics. The modern theory of conservation law must be formulated in terms of quantum field theory. Harmonic oscillators play a fundamental role in quantum mechanics mainly because the theory of the theory of harmonic oscillator underlies the formal apparatus of second quantization. Lie algebras arise naturally as current in quantum field theory.

We begin by discussing harmonic oscillators. The unitary groups arise naturally in the discussion of harmonic oscillators. We adopt the quantum-mechanical point of view because it is simpler , but the same type of discussion can be given in classical mechanics. The Hamiltonian for a system of n identical non interacting harmonic oscillators is

$$H = \sum_{\alpha=1}^{n} \left\{ \frac{1}{2m} p_\alpha{}^2 + \frac{k}{2} q_\alpha{}^2 \right\}$$

For convenience we choose units in which $m = k = 1$ as well as $h = 2\pi$.

We define creation and annihilation operators by

$$a_\alpha = \frac{1}{\sqrt{2}} (q_\alpha + ip_\alpha) , \quad a_\alpha{}^* = \frac{1}{\sqrt{2}} (q_\alpha - ip_\alpha) ,$$

In terms of which the Hamiltonian may be written as

$$H = (\text{ const. }) + \sum_{\alpha=1}^{n} a_\alpha{}^* a_\alpha .$$

The creation and annihilation operators satisfy the commutation relations

$$[a_\alpha , a_\beta] = 0 , \quad [a_\alpha , a_\beta{}^*] = \delta_{\alpha\beta} , \quad [a_\alpha{}^* , a_\beta{}^*] = 0 ,$$

with each other, and

$$[H, a_\alpha{}^*] = a_\alpha{}^* , \quad [H, a_\alpha] = -a_\alpha ,$$

with the Hamiltonian. These latter commutation relations imply that if ψ is an eigenvector of H with eigenvalue E, then $a_\alpha\psi$ is an eigenvector with eigenvalue $E - 1$, and $a_\alpha{}^*\psi$ is an eigenvector with eigenvalue $E + 1$.

Both the Hamiltonian and the commutation relations are invariant under a unitary transformation,

$$a_\alpha{}' = \sum_\beta U_{\alpha\beta} a_\beta ,$$

so $U(n)$ is a symmetry group of the n-dimensional harmonic oscillator. The corresponding Lie algebra $u(n)$ is generated by the elements $a_\alpha{}^* a_\beta$,which commute with the Hamiltonian H.

We now discuss the conservation of currents. In quantum field theory one introduces for each type of particle a set of *creation and annihilation operators* on a Hilbert space. If ψ is a certain state (vector in Hilbert space), then the state $a_j{}^*\psi$ is another state which differs from the original state by the addition of a single particle of type j. Similarly, $a_j\psi$ is the state obtained by removing a single particle of type j .

Let a_1 ,a_n denote a set of annihilation operators; their Hermitian conjugates $a_1{}^*$,...,$a_n{}^*$ are creation operators. If we limit discussion to half integer spin particles (fermions) , then these operators satisfy the anticommutation relations,

$$\{ a_i, a_j \} = 0 , \qquad \{a_i , a_j{}^* \} = \delta_{ij} , \qquad \{ a_i{}^* ,a_j{}^* \} = 0 ,$$

where $\{x , y\} = xy + yx$. The physical significance of these relations is the Pauli *exclusion principle*: no two fermions can occupy the same quantum state. Similarly , if the creation and annihilation operators refer to integer spin particles (bosons) then these operators satisfy commutation relations which are identical to the commutation relations written down in the case of a harmonic oscillator.

One can treat the case of fermions and the case of bosons simultaneously. In either case one can verify the commutation relations,

$$[a_i{}^*a_j , a_k{}^*a_l] = \delta_{jk}a_i{}^*a_l - \delta_{il}a_k{}^*a_j ,$$

which form the basis for our discussion. These relations show that the elements $a_i{}^*a_j$ always span a Lie algebra, called the *Lie algebra of currents*

Any operator of the form

$$J = \sum_{i,j} a_i{}^*c_{ij}a_j ,$$

where the c_{ij} are complex numbers, will be called a (generalized) *current*. One reason for this name is that ordinary electric current corresponds to such an operator in quantum field theory .There are , however, many other currents, for example, the total spin of a many- electron system and the total kinetic energy of a system of particles. A current J which does not depend explicitly on the time is said to be a *conserved current* if it commutes with the Hamiltonian, $[H, J] = 0$. The set of conserved currents forms a Lie algebra. It is just this Lie algebra which one studies, for instance, in elementary particle physics.

The formula for the current defines a homomorphism from any Lie algebra of $n \times n$ matrices to the Lie algebra of currents. Let us write the definition above in matrix notation as $J = a^* C a$. If K is another current , say $K = a^* D a$, then $[J, K] = a^*[C, D]a$ follows directly from the commutation relations written down for $a_i^* a_j$ with $a_k^* a_l$, so the mapping h:$C \to a^* C a$ is a homomorphism.

5.2. Lie algebras and special functions

A considerable unification of the theory of the special functions can be achieved by the use of Lie algebraic ideas. The types of functions that can be considered include Bessel functions, Hermite polynomials, parabolic cylinder functions, and Legendre polynomials. A systematic catalog of these functions has been given in a paper by Infield and Hull. The method used by these authors involved factorization of the appropriate second order differential equation. For the Hermite polynomials (related to harmonic oscillators) this leads to the creation and annihilation operators .

Infield and Hull were able to reduce the entire list of possibilities to six overlapping classes. More recently, Miller has shown that all of these six classes arise naturally in the representation theory of four particular Lie algebras, namely, the Lie algebras of the three-dimensional rotation group, the group of Euclidean motions in a plane, the Euclidean group in three-dimensional space, and a certain four-dimensional solvable Lie algebra.

We illustrate the general technique by a simple example showing how the theory of Bessel functions is related to the Lie algebra of the group of Euclidean motions in a plane. The Lie algebra of this group is three-dimensional. Any motion can be composed of a rotation and a translation, the angle ϕ of the rotation and the orthogonal components (x, y) of the translation give local coordinates for a neighbourhood of the identity. The tangent vectors corresponding to the subgroup of translations in the x direction will be denoted $\mathbf{T}_x$, that corresponding to translations in the y direction $\mathbf{T}_y$, while the tangent vector to the subgroup of rotations will be $\mathbf{L}$. The vectors $\mathbf{T}_x$, $\mathbf{T}_y$, $\mathbf{L}$ are a basis for the Lie algebra.

Setting

$$\mathbf{T}_x = \frac{\partial}{\partial x}\,, \quad \mathbf{T}_y = \frac{\partial}{\partial y}\,, \quad \mathbf{L} = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) = -i\frac{\partial}{\partial \phi}$$

yields an isomorphism of the Lie algebra of the group with a Lie algebra of operators on the infinitely differentiable functions on the plane, the Lie multiplication being, as usual, commutation. If we define

$$\mathbf{T}_{\pm} = \mathbf{T}_x \pm i\,\mathbf{T}_y = \frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y} \quad = \exp(\pm i\phi)\left(\frac{\partial}{\partial r} \pm \frac{i}{r}\frac{\partial}{\partial \phi}\right),$$

then we find that the commutation relations take the particularity simple form,

$$[T_+ , T_-] = 0 \quad , \qquad [L , T_\pm] = \pm T_\pm .$$

Many other properties of Bessel functions can be obtained by use of the Lie algebra.

5.3. Physical applications

5.3.1 Isospin and $SU(2)$

The simplest case of an application in physics can be found in a Lie algebra generated from the bilinear products of creation and annihilation operators where there are only two quantum states. This is often referred to as the "old fashioned" isospin as it was originally conceived for systems of neutrons and protons before the discovery of mesons and strange particles. The concept of isospin was first introduced by Heisenberg in 1932 to explain the symmetries of newly discovered neutrons. Although the proton has a positive charge, and neutron is neutral, they are almost identical in other respects such as their masses. Hence, the term `nucleon' was coined: treating two particles as two different states of the same particle, the nucleon. In fact, the strength of strong interaction – the force which is responsible forming the nucleus of an atom - between any pair of nucleons is independent of whether they are interacting as protons and nucleons. More precisely, the isospin symmetry is given by the invariance of Hamiltonian of the strong interactions under the action of Lie group, $SU(2)$. The neutrons and protons are assigned to the doublets with spin 1/2-representation of "$SU(2)$. Let us take a more detailed look in the mathematical formulation.

Let $a_p^\dagger$ and $a_n^\dagger$ be operators for the creation of a proton and neutron, respectively, and let a_p and a_n be the corresponding annihilation operators. Now, construct the Lie algebra of all possible bilinear products of these operators which do not change the number of particles (strong interaction invariance). There are four possible bilinear products:

$$a_p^\dagger a_n\,, \quad a_n^\dagger a_p\,, \quad a_p^\dagger a_p\,, \quad a_p^\dagger a_p$$

The first operator turns a neutron into a proton, while the second operator turns a proton into a neutron. Let us denote the first two operators by T_+ and T_-. Recall, this whole symmetry is based on the idea that the proton and the neutron are simply two different states of the same particle: we can treat the proton as having spin-up and the neutron as having spin-down, i.e., associating them with doublets $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. With that in mind, the notation would seem more natural: T_+ being the raising operator, while T_- being the lowering operator. Now, the last two operators simply annihilates a proton or neutron and then create them back. These are just the number operators which count the number of protons and neutrons. Together, they are total number operators, which commute with the all the other operators, as they do not change the total number. It is therefore convenient to divide the set of four operators into a set of three operators plus the total number or Baryon number, operator, which commutes with others:

$$B = a_p^\dagger a_p + a_n^\dagger a_n\,,$$

$$\mathrm{T}_+ = a_p^\dagger\, a_n$$

$$T_- = a_n^\dagger a_p$$

$$T_0 = \frac{1}{2}(a_p^\dagger a_p - a_n^\dagger a_n) = Q - \frac{1}{2}B$$

where Q is just the total charge (Since proton has a positive charge whereas neutron has no charge.). Now, the set of three operators , T_+, T_-, T_0 satisfy following commutation relation, which is exactly like that of angular momenta:

$$[T_0, T_+] = T_+,$$

$$[T_0, T_-] = -T_-$$

$$[T_+, T_-] = 2T_0$$

This has led to the designation isospin for these operators and to the description of rotations in the fictitious isospin space.

Let us now consider which Lie group is associated with these isospin operators. By analogy with angular momentum operators, we allow these operators to generate infinitesimal transformations such as

$$\psi' = \{1 + i\epsilon(T_+ + T_-)\}\psi$$

We use the linear combination $T_+ + T_-$ because these are not individually Hermitian(or self-adjoint). Note that such a transformation changes a proton or neutron into something which is a linear combination of the proton and the neutron state. These transformations are thus transformations in a two-dimensional proton-neutron Hilbert space. The transformations are unitary thus the Lie group associated with isospin is some group of unitary transformation in a two-dimensional space. The whole group of unitary transformations in a two-dimensional space is generated by the set of four operators; however, the unitary transformations generated by the operator B

are of a trivial nature: they are multiplication of any state by a phase factor. Since the three isospin operators form a Lie group by themselves, the associated group is the subgroup of full unitary group in two-dimensions, the special unitary group, $SU(2)$

5.3.2. Eightfold way and $SU(3)$

As noted before, the above isospin $SU(2)$ symmetry is old fashioned in that it does not consider mesons and the "strangeness" - a property in particles, expressed as quantum numbers, for describing a decay of particles in strong and electromagnetic interactions, which occur in a short amount of time. This was first introduced by Murray Gell-Mann and Kazuhiko Nijishima to explain that certain particles such as the kaons or certain hyperons were created easily in particle collisions, yet decayed much more slowly than expected for their large masses. To account for the newly added property (quantum number), the Lie group $SU(3)$ was chosen over $SU(2)$ to construct a theory which organizes baryons and mesons into octets (thus, the term Eightfold Way), where the octets are the representations of the Lie group, $SU(3)$.

CONCLUSION

Lie algebra arises naturally in the study of mathematical objects called Lie groups, which serve as groups of transformations on spaces with certain symmetries. We also introduced the basic properties and some algebraic facts related to Lie algebra.

In this project we gave a collection of typical examples of Lie algebra and introduced the basic vocabulary of Lie algebras. We also discussed about the semisimple Lie algebra which has a vital role in the study of Lie algebra. An arbitrary Lie algebra is a semidirect sum of a semisimple Lie algebra and a solvable invariant subalgebra. The structure of Lie algebra can be determined by inspecting its regular representation, once this has been brought to suitable form by a similarity transformation. Here we introduced the representation of a semisimple Lie algebra interms of its rootspace decomposition. The essential properties of the roots of complex semisimple Lie algebras may be captured in the idea of an abstract " root system ". In this project we developed the basic theory of root system.

Lie algebra play a fundamental role in modern mathematics and physics. Since it has a wide range of applications only few of them are discussed here. Symmetries in physics take on the structure of mathematical groups, where continuous groups represent Lie groups. Collisions of vector bosons in quantum field theory is tied to Lie algebra, specifically gauge theory, and as such, becomes a mechanism to interpolate physical interactions. The algebra can be promoted to a group and interpreted as a symmetry. Lie algebras are thus integral to describing theories of nature.

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