LINEAR FRACTIONAL PROGRAMMING

PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR

THE MASTER DEGREE IN MATHEMATICS

BY

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CERTIFICATE

This is to certify that the project entitled **" LINEAR FRACTIONAL PROGRAMMING "** is a bonafide record of studies undertaken by ANITTA JOSE (Reg no. 180011015178), in partial fulfillment of the requirements for the award of M.Sc. Degree in Mathematics at Department of Mathematics, St. Paul's College, Kalamassery, during 2018 – 20.

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Examiner :

DECLARATION

I **ANITTA JOSE** hereby declare that the project entitled **"LINEAR FRACTIONAL PROGRAMMING"** submitted to department of Mathematics St. Paul's College, Kalamassery in partial requirement for the award of M.Sc Degree in Mathematics, is a work done by me under the guidance and supervision of **Miss Maya .K,** Department of Mathematics, St. Pauls's College , Kalamassery during 2018 – 20.

I also declare that this project has not been previously presented for the award of any other degree, diploma, fellowship, etc.

Kalamassery ANITTA JOSE

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Kalamassery **Anitta Jose**

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INTRODUCTION

Mathematical programming has know a spectacular diversification in the last few decades. This process has happened at the level of mathematical research and at the level of application generated by the solution method that were created. The field of LFP, largely developed by Hungarian mathematician B.Martos and his associates in the 1960 s is concerned with problems of optimization. Linear fractional programming problems deals with determining the best possible allocation of available resources to meet certain specifications. In particular, that may deal with situation where a number of resources, such as people, materials, machines and land available and are to be combined to yield several products. In linear fractional programming, the goal is to determine a permissible allocation of resources that will maximize or minimize some specific showing such as profit gained per unit of cost or cost of unit of product produced etc.

Interest of this subject was generated by the fact that various optimization problems from engineering and economics consider the minimization of a ratio between physical and economical functions. For example, cost/time, cost/volume, cost/profit or other quantities that measure the efficiency of a system. For example the productivity of industrial system , defined as the ration between the realized services in a system within a given period of time and the utilized resources , is used as one of the best indicators of the quality of the operation such problems where the objective function appears as a ration of functions constitute fractional programming problem.

Strictly speaking, linear fractional programming is a special case of the boarder field of mathematical programming. Linear fractional programming deals with that class of mathematical programming problems in which the relations among the variables are linear. The constraint relations must be in linear form and the function to be optimized must be a ratio of two linear functions. At the same time linear fractional programming includes as a special case the well known and wide spread. Linear Programming (LP.) In the problems of LP both the restrictions and the objective function must be linear in form. If in an LFP problem the denominator of the objective function is constant, which equals to 1 then we have an LP problem. Conversely any problem of LP may be considered as an LFP one with the constant denominator of the objective function. Due to its importance in mode various decision process in management science, operational research and economics and also due to its frequent appearance in other problems that are not necessarily economical such as information theory of numerical analysis, stochastic programming, decomposition algorithms for large linear systems etc. The fractional programming method has received particular attention in the last three decades

This project deals with linear-fractional programming (LFP). The object of LFP is to find the optimal (maximal or minimal) value of a linear fractional objective function subject to linear constraints on the given variables. If all unknown variables are real valued then we say that the problem is real or continuous. In the case of one or more integer-valued variables we usually say that the problem is integer or IP. The IP problem may be pure, if all the variables must have in optimal solution an integer value, or mixed in the other case. The constraints in the problem may be either equality or inequality constraints1. From the point of view of real-world applications, LFP possesses as many nice and extremely useful features, as linear programming (LP). If we have a problem formulated as an LP one, we can re-formulate this problem as LFP by replacing an original linear objective function with a ratio (fraction) of two linear functions. If in the original LP problem the objective function expresses, for example, the profit of some company, in the case of the LFP problem we can optimize the activity of the company in accordance with such fractional criteria as profit/cost or profit/manpower requirement and so on. Moreover, from the point of view of applications such an optimal solution is often more preferable and attractive than obtained from the LP problem because of higher efficiency. Problems of LFP arise when there appear a necessity to optimize the efficiency of some activity: profit gained by company per unit of expenditure of labor, cost of production per unit of produced goods , nutritiousness of ration per unit of cost, etc. Nowadays because of a deflect of natural resources the use of such specific criteria becomes more and more topical and relevant. So an application of LFP to solving real-world problems connected with optimizing efficiency could be as useful as in the case of LP. The only problem is that until now there has been no well-made software package developed especially for using LFP and teaching it.

CHAPTER-1

INTRODUCTION TO LFP

LINEAR-FRACTIONAL PROBLEM

In 1960, Hungarian mathematician Bela Martos formulated and considered a so-called hyperbolic programming problem, which in the English language special literature is referred as a linear-fractional programming problem. In a typical case the common problem of LFP may be formulated as follows:

Given objective function

$$Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{j=1}^{n} p_j x_j + p_0}{\sum_{j=1}^{n} d_j x_j + d_0} \qquad \_\_\_\_\_\_ (1.1)$$

which must be maximized (or minimized) subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1,2 \ldots\ldots\ldots m_1 \qquad (1.2\ a)$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \qquad i = m_1 + 1, m_1 + 2, \ldots\ldots m_2 \qquad (1.2\ b)$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = m_2 + 1, m_2 + 2, \ldots\ldots m \qquad (1.2\ c)$$

$$x_j \geq 0,\ j = 1, 2, \ldots\ldots, n_1, \qquad \_\_\_\_\_\_ (1.3)$$

where $m_1 \leq m_2 \leq m$, $n_1 \leq n$ Here and in what follows we suppose that $D(x) \neq 0$, $\forall x = (x_1, x_2, \ldots . x_n) \in S$, where S denotes a feasible set of solutions defined by constraints (1.2), (1.3).

Because denominator $D(x) \neq 0\ \forall x \in S$, without loss of generality we can assume that

$$D(x) > 0\ \forall x \in S \qquad \_\_\_\_ (1.4)$$

In the case of $D(x) < 0$ we can multiply numerator P(x) and denominator $D(x)$ of objective function Q(x) with (-1).

Here and in what follows throughout the book we deal with just such linear-fractional programming problems that satisfy condition (1.4). Furthermore, we suppose that all constraints in system (1.2) are linearly independent and so the rank of matrix A = $\|a_{ij}\|_{m\times n}$ is equal to m. So in an LFP problem our aim is to find such a vector x of decision variables x_j, j=1,... n, which

1. maximizes (or minimizes) function Q(x), called objective function, and at the same time
2. satisfies a set of main constraints (1.2) and sign restrictions (1.3).

1.1 Main Definitions

Here we introduce the main conceptions that will be used throughout the rest of the book.

DEFINITION 1.1 :

If given vector x = $(x_1, x_2, \dots x_n)$ satisfies constraints (1 .2) and (1.3), we will say that vector x is a feasible solution of LFP problem (1.1)-(1.3).

DEFINITION 1.2:

If given vector x = $(x_1, x_2, \dots . x_n)$ is a feasible solution of maximization (minimization) LFP problem (1.1)-(1.3), and provides maximal (minimal) value for objective function Q(x) over the feasible set S, we say that vector x is an optimal solution of maximization (minimization) linear-fractional programming problem (1.1)-(1.3).

DEFINITION 1.3 :

We say that a maximization (minimization) linear-fractional programming problem is solvable, if its feasible set S is not empty, that is S $\neq 0$, and objective function Q(x) has finite upper (lower) bound on S.

DEFINITION 1.4 :

If the feasible set is empty, that is S = 0, we say that the LFP problem is infeasible.

DEFINITION 1.5 :

If objective function Q(x) of a maximization (minimization) LFP problem has no upper (lower) finite bound, we say that the problem is unbounded

1.2: Relationship with Linear Programming

If all $d_j = 0$, j = 1, 2, ... n, and $d_0 = 1$, then LFP problem becomes an LP problem. This is a reason why we say that an LFP problem is a generalization of an LP problem:

Given objective function

P(x) = $\sum_{j=1}^{n} p_j\, x_j + p_0$ _______ (1.5)

which must be maximized (or minimized) subject to

$\sum_{j=1}^{n} a_{ij}x_j \leq b_i,$ i = 1, 2,...., m_1, _____ (1.6a)

$\sum_{j=1}^{n} a_{ij}x_j \geq b_i,$ i = m_1+1, m_2+2,…, m_2, _____ (1.6b)

$\sum_{j=1}^{n} a_{ij}x_j = b_i,$ i = m_2+1, m_2+2,…., m _____ (1.6c)

$x_j \geq 0,$ j = 1,2, ... ,n_1, _____ (1.7)

There are also a few special cases when the original LFP problem may be replaced with an appropriate LFP problem:

.

1. If $d_j = 0$, j = 1, 2, ... n, $d_0 \neq 0$, then objective function Q(x) becomes a linear one:

Q(x) = $\sum_{j=1}^{n} \frac{p_j}{d_0} x_j + \frac{p_0}{d_0} = \frac{p(x)}{d_0}$ _______ (1.8)

In this case maximization (minimization) of the original objective function Q(x) may be substituted with maximization (minimization) of linear function P(x)/d_0 correspondingly on the same feasible set S.

2. If p_j = 0, j = 1, 2, ... n, then objective function

$$Q(x) = \frac{P(x)}{D(x)} = \frac{p_0}{\sum_{j=1}^{n} d_j x_j + d_0} \qquad \text{\_\_\_ (1.9)}$$

may be replaced with function D(x). In this case maximization (minimization) of the original objective function Q(x) must be substituted with minimization (maximization) of a new objective function D(x) on the same feasible set S.

3. If vectors p = $(p_1, p_2, \dots, p_n)$ and d = $(d_1, d_2, \dots, d_n)$ are linearly dependent, that is there exists such $\mu \neq 0$ and $p = \mu d$, then objective function

$$Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{j=1}^{n} \mu d_j x_j + p_0}{\sum_{j=1}^{n} d_j x_j + d_0} = \dots = \mu + \frac{p_0 - \mu d_0}{\sum_{j=1}^{n} d_j x_j + d_0} \qquad \text{\_\_(1.10)}$$

may be replaced with function D(x). Obviously, in this case maximization (minimization) of the original objective function Q(x) must be substituted with

- minimization (maximization) of D(x), if

$p_0 - \mu d_0 > 0$,

- maximization (minimization) of D(x), if

$p_0 - \mu d_0 < 0$

____ (1.11)

We should note here that in the case of $p_0 - \mu d_0 = 0$, we have Q(x) = μ which means that Q(x) = constant, $\forall x \in$ S We will not consider such problems because of their pointlessness.

Here we exclude from our consideration the following three trivial cases:

1. P(x) = constant, $\forall x \in$ S;

2. D(x) = constant, $\forall x \in$ S; _____ (1.12)

3. Q(x) = constant, $\forall x \in$ S;

because in these cases the original LFP problem may be reduced to an LP problem (first two cases), or becomes absolutely aimless (case 3).

1.3 Main Forms of the LFP Problem

We have seen that LFP problems may have both equality and inequality constraints. They may also have unknown variables that are required to be nonnegative and variables that are allowed to be unrestricted in sign (urs variable). Before the simplex method is discussed we should introduce some special forms of formulating an LFP problem and show how these forms may be converted to one another and to the form that is required by simplex method.

DEFINITION : An LFP problem is said to be in **standard form** if all constraints are equations and all unknown variables are non-negative, that is

Max(min): $Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{j=1}^{n} p_j x_j + p_0}{\sum_{j=1}^{n} d_j x_j + d_0}$ ____ (1.13)

Subject to: $\sum_{j=1}^{n} a_{ij} x_j = b_i$, i =1,2,…,m, ____ (1.14)

$x_j \geq 0$, j = 1,2,…,n, _____ (1.15)

Where $D(x) > 0, \forall x \in$ S.

DEFINITION : An LFP problem is said to be in **general form** if all constraints are ≤ ('less than') inequalities and all unknown variables are non-negative, that is

Max(min): $Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{j=1}^{n} p_j x_j + p_0}{\sum_{j=1}^{n} d_j x_j + d_o}$ _____ (1.16)

Subject to: $\sum_{j=1}^{n} a_{ij}x_j \le b_i$, i = 1,2,…m,

$x_j \ge 0$, j = 1,2,…,n, ____ (1.17)

Where D(x) > 0, $\forall x \in S$

It is obvious that standard and general forms of LFP problems are special cases of a LFP problem formulated in form (1.1) - (1.3). Indeed, if in the common LFP problem (1.1) - (1.3) we put $m_1 = m_2 = 0$ and $n_1 = n$, then we get a standard LFP problem. But if $m_1 = m$ and $n_1 = n$, then we have a general LFP problem.

To convert one form to another we should use the following converting procedures:

1. ' ≥ '('greater than') → ' ≤ ' ('less than').

Both sides of the ' ≥ ' constraint must be multiplied by (-1)

2. ' ≤ ' ('less than') → ' = ' ('equal').

Define for ≤ constraint a non-negative slack variable S_i ($S_i \ge 0$- slack variable for i-th constraint) and put this variable into the left-side of the constraint, where it will play a role of difference between the left and right sides of the original i-th constraint. Also add the sign restrictions $S_i \ge 0$ to the set of constraints. So

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \rightarrow \begin{Bmatrix} \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i \\ s_i \ge 0 \end{Bmatrix} \text{\_\_\_\_ } (1.18)$$

3. Unrestricted in sign variable x_i → restricted in sign non-negative . variable(s) For each urs variable x_j, we begin by defining two new non-negative variables x_j' and x_j'' .Then substitute $x_j' - x_j''$ for x_j in each constraint and in objective function. Also add the sign restrictions $x_j' \ge 0$ and $x_j'' \ge 0$ to the set of constraints

Let us introduce the following notations:

A_J = $(a_{1j}, a_{2j}, \dots . a_{mj})^T$, j = 1, 2, ... , n;

b = $(b_1, b_2, \dots, b_m)^T$, A= (A_1, A_2,...,A_n),

x = $(x_1, x_2, \dots, x_n)^T$, p = $(p_1, p_2, \dots, p_n)^T$, d =$(d_1, d_2, \dots, d_n)^T$.

Using this notation we can re-formulate an LFP problem in a matrix form:

Standard problem

$$Q(x) = \frac{p^T x + p_0}{d^T x + d_0} \rightarrow max,$$

subject to : $\sum_{j=1}^{n} A_j x_j = b,$

$$x \geq 0,$$

Where $D(x) = d^T x + d_0 > 0, \forall x \in S$

General Problem

$$Q(x) = \frac{p^T x + p_0}{d^T x + d_0} \rightarrow max$$

subject to : $Ax \leq b,$

$$x \geq 0,$$

Where $D(x) = d^T x + d_0 > 0, \forall x \in S$

We should note here that in accordance with the theory of mathematical programming

$$\min_{x \in S} F(x) \equiv \max_{x \in S}(-F(x))$$

CHAPTER – 2

THE GRAPHICAL METHOD

We now go on to discuss how any LFP problem with only two variables can be solved graphically. Consider the following LFP problem with two unknown variables:

$$Q(x) = \frac{P(x)}{D(x)} = \frac{p_1x_1+p_2x_2+p_0}{d_1x_1+d_2x_2+d_0} \rightarrow \max(min) \quad (2.1)$$

Subject to : $a_{i1}x_1 + a_{i2}x_2 \leq b_i$ (2.2)

$x_1 \geq 0, x_2 \geq 0.$ (2.3)

2.1 The Single Optimal Vertex

Let us suppose that constraints (2.2) and (2.3) define feasible set S shown by shading in Figure 2.11. Let Q(x) = K, where K is an arbitrary constant

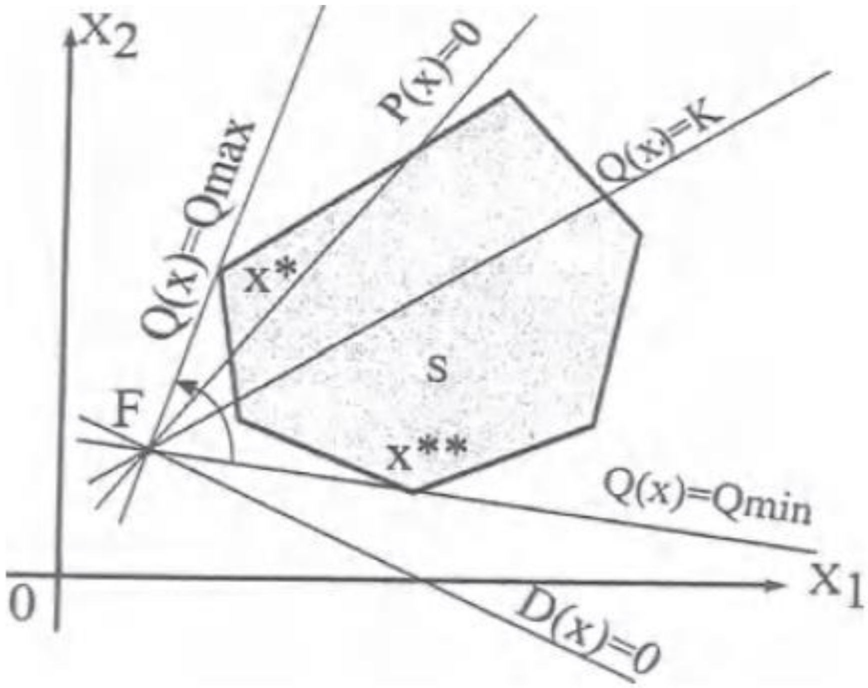

Figure 2.11. Two-variable LFP problem-Single optimal vertex

For any real value K, equation

Q(x) = K

Or $(p_1 - Kd_1)x_1$+$(p_2 - Kd_2)x_2$ + $(p_0 - kd_0)$ = 0

represents all the points on a straight line in the two-dimensional plane x1Ox2. If this so-called level-line (or isoline) intersects the set of feasible solutions S, the points of intersection are the feasible solutions that give the value K to the objective function Q(x). Changing the value of K translates the entire line to another line that intersects the previous line in focus point (point F in Figure 2.11) with coordinates defined as solution of system

$p_1x_1 + p_2x_2 = -p_0$ ____ (2.4)

$d_1x_1 + p2x_2 = -d_0$

In other words, in the focus point F straight lines with equations P(x) = 0 and D(x) = 0 intersect one another.

If lines P(x) = 0 and D(x) = 0 are not parallel with one another, then the determinant of system (2.4) is not equal to zero and the system has a unique solution (coordinates of focus point F). In the other case, if lines P(x) = 0 and D(x) = 0 are parallel with one another, the determinant of system (2.3) is equal to zero and the system has no solution. It means that there is no focus point and all level-lines are also parallel with one another. The given LFP problem (2.1)-(2.2) degenerates to an LP one. Hence, to maximize objective function Q(x) we should minimize or maximize its denominator D(x) depending on the sign of expression

$p_0 - \mu d_0$.

Let us return to the case when level-lines are not parallel with one another. Pick an arbitrary value of K and draw the line Q(x) = K as follows.

$$x_2 = \frac{p_1 - kd_1}{p_2 - kd_2} x_1 - \frac{p_0 - kd_0}{p_2 - kd_2}$$

In such a case the slope

$$k = -\frac{p_1 - kd_1}{p_2 - kd_2}$$

of level-line Q(x) = K depends on value K of objective function Q(x), and is a monotonic function on K because

$$\frac{dk}{dK} = \frac{d_1 p_2 - d_2 p_1}{(p_2 - Kd_2)^2}$$

Further, the sign of $\frac{dk}{dK}$ does not depend on the value of K, so we can write

sign { $\frac{dk}{dK}$} = sign {$d_1p_2 - d_2p_1$} = const.

It means that when rotating level-line around its focus point F in positive direction (i.e.counter clockwise), the value of objective function Q(x) increases or decreases depending on the sign of expression ($d_1p_2 - d_2p_1$). Obviously, Figure 2.11 represents the case when rotating level-line in positive direction leads to growth of value Q(x). When rotating level-line around its focus point F the line Q(x) = K intersects feasible set S in two vertices (extreme points) x* and x**. In the point x* objective function Q(x) takes its maximal value over set S and in the point x** it takes its minimal value.

2.2 Multiple Optimal Solutions

It may occur that when rotating level-line on its focus point F the level-line

Q(x) = K captures some edge of set S (see edge e in Figure 2.12). In this case

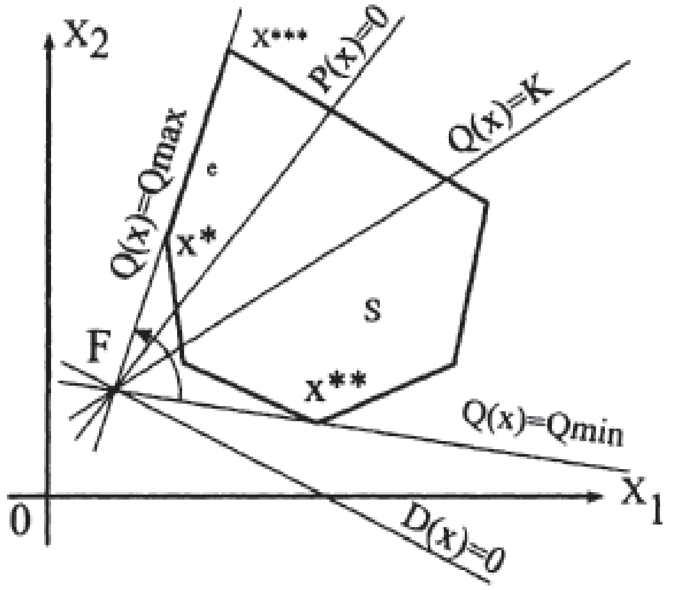

Figure 2.12. Two-variable LFP problem-Multiple optimal solutions.

the problem has an infinite number of optimal solutions (all points x of edge e) that may be represented as a linear combination of two vertex points x* and x***:

$x = \lambda x^{*} + (1-\lambda)x^{***}, 0 \le \lambda \le 1$

2.3 Mixed cases

If feasible set S is unbounded and an appropriate unbounded edge concurs with extreme level-line (see Figure 2.13), then the problem has an infinite number of optimal solutions one of them in vertex x* and others are non-vertex points unbounded edge. We should note here that among these non-vertex points there is one infinite point too. This is why we say in this case that the problem has 'mixed' solutions, i.e. finite optimal solution(s) and asymptotic one(s).

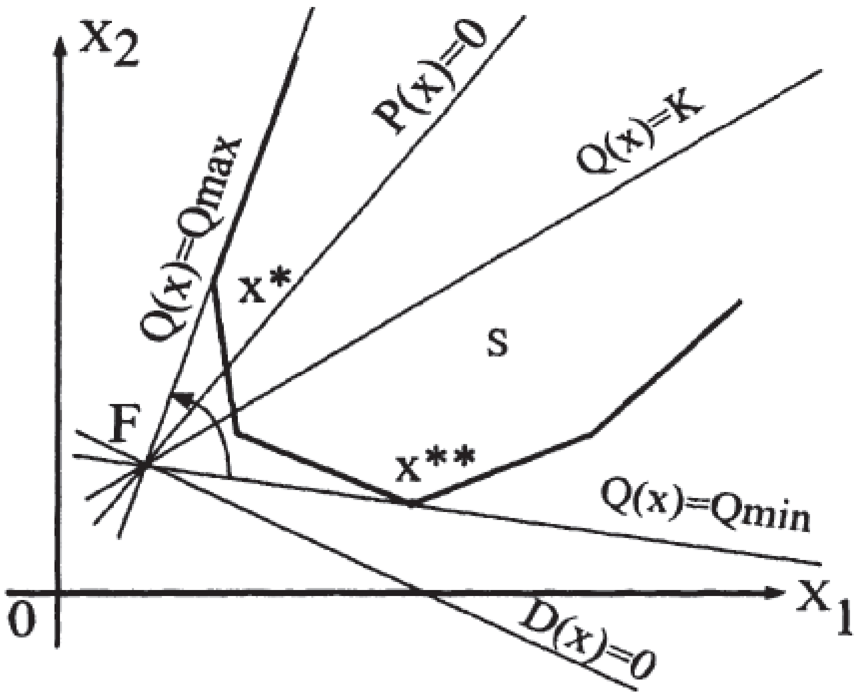

Figure 2.13. Two-variable LFP problem - Mixed case.

CHAPTER 3

Methods For Solving Linear Fractional Programming Problems

3.1 Charnes & Cooper's Transformation

In 1962 A. Charnes and W.W. Cooper showed that any linear-fractional programming problem with a bounded set of feasible solutions may be converted to a linear programming problem. Consider the common LFP problem (1.1) - (1.3). Let us introduce the following new t_j variables:

$$t_j = \frac{x_j}{D(x)}, \ j = 1,2,\ldots,n, \quad t_0 = \frac{1}{D(x)}$$

Where,

$$D(x) = \sum_{j=1}^{n} d_j x_j + d_0$$

Using these new variables t_j , $j = 0,1,\ldots,n$, we can rewrite the original objective function Q(x) in the following form

$L(t) = \sum p_j t_j \ \rightarrow \max(or\ min)$ ____(3.2)

Since we suppose that $D(x) > 0 \forall\ x \ \in S$, we can multiply all constraints of (1.2) and (1.3) by 1/D(x), so we obtain the following constraints:

$$-b_i t_0 + \sum_{j=1}^{n} a_{ij}\, t_j \leq 0, \quad i = 1,2,\ldots,m_1$$

$$-b_i t_0 + \sum_{j=1}^{n} a_{ij}\, t_j \geq 0, i = m_1+1, m_1+2,\ldots,m_2 \quad \text{\_\_\_(3.3)}$$

$$-b_i t_0 + \sum_{j=1}^{n} a_{ij} t_j = 0, i = m_2+1, m_2+2,\ldots,m$$

$$t_j \geq 0; \ \ j = 0,1,2,\ldots,m_1 \qquad \text{\_\_\_\_\_\_(3.4)}$$

The connection between the original variables x_j and the new variables t_j will be completed if we multiply equality (3.1) by the same value 1/D(x), and then append the new constraint to the new problem:

$\sum_{j=1}^{n} d_j t_j = 1$ ____(3.5)

Here and in what follows the new problem (3.2)-(3.5) will be referred to as a linear analogue of an LFP problem. This transformation (usually referred to as Charnes and Cooper transformation) of variables establishes a one ↔ one connection between the original LFP problem and its linear analogue

Since feasible set S is bounded, function D(x) is linear and D(x) > 0, $\forall x \in S$, the following statement may be formulated and proved:

THEOREM : 3.1

If vector $t^* = (t^*_0, t^*_1, \ldots, t^*_n)^T$ optimal solution of problem (3.2)-(3.5), then vector $x^* = (x_0^*, x_1^*, \ldots, x_n^*)^T$ is an optimal solution of original LFP problem (1.1)-(1.3), where

$$x_j^* = \frac{t^*_j}{t^*_j}, j = 1,2, \ldots n$$

Proof.

We prove this statement only for the case of maximization problems. In the case of minimization the proof may be implemented in an analogous way. Since vector t* is the optimal solution of maximization linear analogue, it follows that

$$L(t^*) \geq L(t), \forall_t \in T,$$

where T denotes a feasible set of linear analogue Let us suppose that vector x* is not an optimal solution of the maximization LFP problem. Hence, there exist some another vector x' ∈ S, such that Q(x') ≥ Q(x*). But at the same time

$$Q(x^*) = \frac{\sum_{j=1}^{n} p_j x^*_j + p_0}{\sum_{j=1}^{n} d_j x^*_j + d_0} = \frac{\sum_{j=1}^{n} p_j \frac{t^*_j}{t^*_0} + p_0}{\sum_{j=1}^{n} d_j \frac{t^*_j}{t^*_0} + d_0}$$

$$= \frac{\sum_{j=1}^{n} p_j t^*_j + p_0 t^*_0}{\sum_{j=1}^{n} d_j x^*_j + d_0 t^*_0} = \frac{\sum_{j=1}^{n} p_j t^*_j + p_0 t^*_0}{1} = L(t^*)$$

Since vector x' is a feasible solution of the original LFP problem, it is easy to show that vector

$$t' = (t'_0, t'_1, \dots, t'_n)^T, t'_0 = \frac{1}{D(x')}, t'_j = \frac{x'_j}{D(x')}, j=1,2,\dots n$$

is a feasible solution of linear analogue and

$L(t') \geq L(t^*)$.

But the latter contradicts our assumption that vector $t^*$is an optimal solution of the maximization problem (3.2)-(3.5). It means that vector x^* is an optimal solution of the maximization LFP problem

Example – 1

$$\text{Max } Q(x) = \frac{8x_1+9x_2+4x_3+4}{2x_1+3x_2+2x_3+7}$$

Subject to

$$1x_1 + 1x_2 + 2x_3 \leq 3,$$

$$2x_1 + 1x_2 + 4x_3 \leq 4,$$

$$5x_1 + 3x_2 + 1x_3 \leq 15,$$

$$x_j \geq 0,\ j = 1,2,3$$

Solving this LFP problem we obtain

$$x^* = (1,2,0)^T, p(x^*) = 30, D(x^*) = 15, Q(x^*) = 2$$

we construct the following linear analogue of our LFP problem

$$L(t) = 4t_0 + 8t_1 + 9t_2 + 4t_3 \rightarrow max$$

Subject to

$7t_0 + 2t_1 + 3t_2 + 2t_3 = 1,$

$-3t_0 + 1t_1 + 1t_2 + 2t_3 \leq 0,$

$-4t_0 + 2t_1 + 1t_2 + 4t_3 \leq 0,$

$-15t_0 + 5t_1 + 3t_2 + 1t_3 \leq 0,$

$t_j \geq 0,\ j = 1,2,3$

If we solve this linear programming problem we have So,

$$t^* = (\frac{1}{15}, \frac{1}{15}, \frac{2}{15}, 0)^T, L(t^*) = 2$$

$$t^* = \frac{1}{15},\ t_1^* = \frac{1}{15},\ t_2^* = \frac{2}{15},\ t_3^* = \frac{0}{15}$$

We should note here that in the case of an unbounded feasible set S it may occur that in the optimal solution of the linear analogue t_0^* = 0. It means that the optimal solution of the original LFP problem is asymptotic and the optimal solution x* contains variables with an infinite value. The connection between the optimal solutions of the original LFP problem and its linear analogue formulated in Theorem seems to be very useful and at least from the point of view of theory allows to substitute the original LFP problem with its linear analogue and in this way to use LP theory and methods. However, in practice this approach based on the Charnes and Cooper transformation may not always be utilized. The problems arise when we should transform an LFP problem with some special structure of constraints, for example transportation problem, or assignment problem or any other problem with a fixed structure of constraints, and would like to apply appropriate special methods and algorithms. Indeed, if in the original LFP problem we have n unknown variables and m main conditions, then in its linear analogue we obtain n + 1 variables and m + 1 constraints. Moreover, in the right-hand side of system (3.3) we have no vector b. Instead of the original vector b we have a vector of zeros. which means that we cannot apply the main results of duality theory formulated for LFP problems

All these changes in the structure of constraints means that the use of special methods and algorithms in this case becomes very difficult or absolutely impossible. This is why, in spite of the existence of the Charnes and Cooper transformation, we will focus on a direct approach to the investigation of an LFP problem and as we have seen the use of such a direct approach is necessary and unavoidable

Dinkelbach's Algorithm

One of the most popular and general strategies for fractional programming (not necessary linear) is the parametric approach described by W.Dinkelbach . In the case of linear-fractional programming this method reduces the solution of a problem to the solution of a sequence of linear programming problems.

Consider the common LFP problem (1.1)-(1.3) and function

$$F(\lambda) = \frac{max}{x \in s} \{P(x) - \lambda D(x)\}\,, \lambda \in R$$

where S denotes the feasible set of (1.1)-(1.3).

The following theorem plays the role of the theoretical foundation of Dinkelbach's method.

THEOREM

Vector x^* an optimal solution of the LFP problem (1.1)-(1.3) if and only if

$$F(\lambda^*) = \frac{max}{x \in S}\{P(x) - \lambda^* D(x)\} = 0 \qquad \_\_\_\_\_(3.1)$$

Where, $\lambda^* = \frac{P(x^*)}{D(x^*)}$

Proof. If vector x* is an optimal solution of problem (1.1)-(1.3) then

$$\lambda^* = \frac{P(x)^*}{D(x)^*} \geq \frac{P(x)}{D(x)}, \forall x \in S$$

The latter means that

$$P(x) - \lambda^* D(x) \leq 0, \forall x \in S$$

Taking into account equality we obtain

$$\frac{max}{x \in S}\{P(x) - \lambda^* D(x)\} = 0.$$

Conversely, if vector x* is an optimal solution of problem (3.1)

Then

$$P(x) - \lambda^* \leq P(x^*) - \lambda^* D(x^*) = 0 \ \forall x \in S.$$

This means that vector x* is an optimal solution of LFP problem (1.1)-(1.3). This theorem also gives a procedure for calculating the optimal solution of linear-fractional programming problem

Dinkelbach's Algorithm

Step 0. Take $x^0 \in S$ compute $\lambda^1 = \frac{P(x^0)}{D(x^0)}$, ·= and let k =1;

Step 1. Determine $x^{(k)} := \arg \max_{x \in S}\left\{P(x) - \lambda^{(k)} D(x)\right\}$

Step 2. If F $(\lambda^k) = 0$ then is an optimal solution; Stop;

Step 3. Let $\lambda^{(k+1)} := \frac{P(x^{(k)})}{D(x^{(k)})}$; let k:= k+1; go to step 1;

Algorithm – Dinkelbach`s Algorithm

Example2:

Max:

$$Q(x) = \frac{P(x)}{D(x)} = \frac{x_1+x_2+5}{3x_1+2x_2+15}$$

Subject to

$$3x_1 + x_2 \leq 6,$$

$$3x_1 + 4x_2 \leq 12, \qquad \text{\_\_\_\_\_\_\_\_ (2.1)}$$

$$x_1 \geq 0, x_2 \geq 0$$

Step 0: Since vector x = $(0,0)^T$ satisfies all constraints of the problem, we may take it as a starting point $x^{(0)} \in S$. So, for $x^{(0)} = (0,0)^T$ we obtain

$$\lambda^{(1)} := \frac{P(x^{(0)})}{D(x^{(0)})} = \frac{5}{15} = \frac{1}{3},$$

Step 1: Now, we have to solve the following linear programming problem

$P(x) - \lambda^{(1)}D(x) = P(x) - \frac{1}{3}D(x) = \frac{1}{3}x_2 \rightarrow max$ Subject to constraints,

Solving this problem we obtain

$$x^{(1)} = (0,3)^T, F\left(\lambda^{(1)}\right) = 1$$

Step 2: Since F $(\lambda^{(1)}) \neq 0$ we have to perform

Step 3: We have to calculate

$$\lambda^{(2)} := \frac{P\left(x^{(1)}\right)}{D\left(x^{(1)}\right)} = \frac{1\times3+5}{2\times3+5} = \frac{8}{21},$$

then to put k:=k+1=2 and repeat

Step 1: Solve the following LP Problem

$P(x) - \lambda^{(2)} D(x) = x^{(2)}$

$$= (0{,}3)^T, F\left(\lambda^{(2)}\right) = 0$$

$$= (1 - \tfrac{8}{21} \times 3)x_1 + (1 - \tfrac{8}{21} \times 2)x_2 + (5 - \tfrac{8}{21} \times 15)$$

$$= -\frac{1}{7}x_1 + \frac{5}{21}x_2 - \frac{5}{7} \rightarrow max$$

Subject to constraints, (2.1)

The optimal solution for this problem

$x^{(2)} = (0.3)^T$ = $(0{,}3)^T$, F($\lambda^{(2)}$) = 0

Step 2: Since F($\lambda^{(2)}$) = 0 vector $x^* = x^{(2)}$

is the optimal solution; stop;

In accordance with the algorithm, the optimal solution of our LFP problem is $x^* = (0{,}3)^T$ with optimal objective value $Q(x^*) = 8/12$.

CHAPTER 4

LFP APPLICATION MODELS

The applications of linear programming to various branches of human activity, and especially to economics, are well known. The applications of linear fractional programming are less known, and, until now, less numerous. Of course, the linearity of a problem makes it easier to deal with, and hence leads to its greater popularity. However, not all real-life problems may be adequately described in the frames of linear models. Linear-fractional programming is a branch of nonlinear programming that was introduced only in the early 60's but since the first publications devoted to LFP problems, this branch has attracted the attention of more and more researchers and specialists because there is a broad field of real-world problems, where the use of LFP is more suitable. In this section of the book we set out to consider several problems that may be formulated in the form of LFP problems.

4.1 Main Economic Interpretation

Let a certain company manufacture n different products. Further, let P_J be the profit gained by the company from a unit of the j-th product, be some constant profit gained whose magnitude is independent of the output volume. The manufacturing of one unit of product j costs $d_j$and there is some constant expenditure d_0 whose value does not depend on the production activity of the company and must be paid for in any case, even if the company does not manufacture anything.

Let b_i be the volume of some scarce resource i available to the company and a_{ij} be the expenditure quota of the i-th resource for manufacturing a unit of j-th kind of the product. The company must decide how many units of each product j should be produced if the efficiency calculated as the ratio (total profit)/(total cost) must be maximized. This problem leads us to define decision variables x_j

the unknown output volume of some j-th product, j = 1, 2, ... , n. The company's total profit (including constant profit P_0) may be expressed as

$$P(x) = \sum_{j=1}^{n} p_j x_j + p_0$$

while the total cost of production activity (including constant expenditure 0) is

$$D(x) = \sum_{j=1}^{n} d_j\, x_j + d_0$$

So the company's objective function may be written as

$$Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{j=1}^{n} p_j x_j + p_0}{\sum_{j=1}^{n} d_j x_j + d_0} \rightarrow max$$

The company's main constraints are the following: $\sum_{j=1}^{n} a_{ij} x_j \le b_i$, i = 1, 2,…m

Since unknown variables express the amount of production to be produced, of course, we also require

$x_j \geq 0$, for all j = 1,2, ... , n.

This problem is formulated in general form as an LFP problem with n unknown non-negative variables and m main constraints.

4.2 A Maritime Transportation Problem

Let us suppose that in port A we have to load a ship of limited carrying capacity C with n types of goods and transport these goods to port B. Our aim is to determine how much of each type of goods must be loaded such that the profit gained per unit of transportation cost be maximal. Let U_j be the maximum available quantity of j-th good, and P_j and d_j be the profit gained per unit of this good and cost of its transportation respectively,j = 1, 2,…,n. If w_j denotes the weight of unit of j-th good, and x_j is an unknown variable, which expresses the quantity of j-th good be loaded, the mathematical model of such a problem may be formulated as follows:

$$\frac{\sum_{j=1}^{n} p_j x_j}{\sum_{j=1}^{n} d_j x_j} \to max$$

subject to

$$\sum_{j=1}^{n} w_j x_j \leq C$$

$$0 \leq x_j \leq U_j, \; j=1,2,\ldots,n$$

The problem formulated in this way is an LFP problem with one main constraint and n unknown nonnegative bounded variables.

4.3 A Financial Problem

Suppose that the financial advisor of a university's endowment fund must invest up to \$100, 000 in two types of securities: bond7Stars, paying a dividend of 7%, and stock MaxMay, paying a dividend of 9%. The adviser has been advised that no more than \$30, 000 can be invested in stockMaxMay, while the amount invested in bond 7Stars must be at least twice the amount invested in stock MaxMay. Independent of the amount to be invested, the service of the broker company which serves the adviser costs\$100. How much should be invested in each security to maximize the efficiency of investment?

Let x and y denote the amounts invested in bond 7Stars and stock MaxMay, respectively. We must then have

$$x+y \leq 100000;$$

$$x \geq 2y;$$

$$y \leq 30000;$$

$$Q(x,y) = \frac{R(x,y)}{D(x,y)} = \frac{0.07x+0.09y}{x+y+100} \to max$$

Of course, we also require

$x \geq 0$, and $y \geq 0$

The return to the university is

R(x, y) = 0.07x + 0.09y

while the total amount of investment is as follows

Q(x,y) = $\frac{R(x,y)}{D(x,y)} = \frac{0.07x+0.09y}{x+y+100} \rightarrow max$

subject to

$$x + y \leq 100000;$$

$$x - 2y \geq 0;$$

$$y \leq 30000.$$

$$x \geq 0, \quad y \geq 0.$$

4.4 A Blending Problem

A metal processor wishes to produce at least15 kilograms of a new alloy NA of lead and tin, containing at least 60% of lead and at least 35% of tin. This new product may be sold for$200 per kilogram. There are four different alloys A_1, A_2, A_3, and A_4 available in amount of 12, 15, 16, and 10 kilograms, respectively . These alloys have the percentage compositions and prices per kilogram is shown below:

	A_1	**A_2**	**A_3**	**A_4**
Lead	40%	60%	80%	70%
Tin	60%	40%	20%	30 %
Costs	**$240**	**$180**	**$160**	**$210**

How should the processor blend alloys A1, A2 A3, and A4 to maximize efficiency of the business? In other words, the processor would like to know how many of each alloy must be blended so that the income/cost ratio would be maximal?

First of all, we define variables x_1, x_2, x_3, and x_4 which express the amount of each alloy to be blended. It is obvious, that the total cost of the blend is

$D(x) = 240x_1 + 180x_2 + 160x_3 + 210x_4$,

while the total income expected from the blend produced and sold is

$P(x) = 200(x_1 + x_2 + x_3 + x_4) = 200x_1 + 200x_2 + 200x_3 + 200x_4$.

The explicit conditions of the problem may be expressed as a following system of inequalities

for lead: $\frac{0.4x1 + 0.6x2 + 0.8x3 + 0.7x4}{x_1+x_2+x_3+x_4} \geq 0.60$

for tin: $\frac{0.6x_1+0.4x_2+0.2x_3+0.3x_4}{x_1+x_2+x_3+x_4} \geq 0.35$

which gives us the following system of linear inequalities

$$-0.20x_1 + 0.00x_2 + 0.20x_3 + 0.10x_4 \geq 0,$$

$$0.25x_1 + 0.05x_2 - 0.15x_3 - 0.05x_4 \geq 0,$$

Since the available value of each alloy is limited, we have the following restrictions

$x_1 \leq 12, x_2 \leq 15, x_3 \leq 16, x_1 \leq 10,$

Finally, we have to add to the system the following condition

$$x_1 + x_2 - x_3 - x_4 \geq 15,$$

since the processor wishes to produce at least 15 kilograms of new alloy. Of course, we also require $x_j \geq 0$, j = 1,2,3,4.

Combining objective function Q(x) = P(x)/D(x) with restrictions leads to the following LFP problem with four bounded variables

$$Q(x) = P(x)/D(x) = \frac{200x_1+200x_2+200x_3+200x_4}{240x_1+180x_2+160x_3+210x_4} \to max$$

subject to

$$-0.021x_1 + \quad +0.20x_3 + 0.10x_4 \geq 0,$$

$$0.25x_1 + 0.05x_2 + 0.20x_3 + 0.10x_4 \geq 0,$$

$$x_1 \leq 12,$$

$$x_2 \leq 15,$$

$$x_3 \leq 16,$$

$$x_4 \leq 10,$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

4.5 Product Planning

Suppose that a refrigerator manufacturer is able to produce five types of refrigerator: Lebel 220, Lebel 120, Star 200, Star 160 and Star 250. The manufacturer has an order from dealers to produce 150, 70 and 290 units of Star 200, Star 160 and Star 250 respectively and 240 units without type detailing (that is, they can be of any type). The manufacturer wishes to formulate a production plan that maximizes its profit gained per unit of cost. All necessary resources excluding Freon 12 and TL 16 aren't scarce. The maximal available quantities of Freon 12 and TL 16 are125 and 80 liters respectively. Manufacturing has the following requirements and known data:

	L 200	L 120	S200	S 160	S 250
TL 16 (litre / unit)	0.2	0.13	__	__	__
F 12 (litre / unit)	__	__	0.22	0.21	0.26
Price $ / unit	420.0	365.0	395.0	355.0	450.0
Cost $ / unit)	320.0	290.0	300.0	280.0	340.0

The manufacturer wishes to satisfy given orders and to get maximum profit gained per unit of total cost of production

. Let x_j , $j =$ 1, 2, 3, 4, 5, denotes the unknown quantities of Lebel 200, Lebel 120, Star 200, Star 160 and Star 250 respectively to be produced. The total profit gained by the manufacturer may be expressed in the following form:

P(x) = (420- 320)x_1 + (365-290)x_2 + (395-300)x_3 + (355- 280)x_4 + (450-340)x_5

Obviously, total cost is

D(x) = 320x_1+ 290x_2 + 300x_3 + 280x_4 + 340.x_5

In this case the objective function will be the following:

$$Q(x) = \frac{P(x)}{D(x)} = \frac{420x_1+365x_2+395x_3+355x_4+450x_5}{320x_1+290x_2+300x_3+280x_4+340x_5} \rightarrow max$$

The main constraints of the problem will be:

Freon : $+0.22x_3 + 0.21x_4 + 0.26x_5 \leq 125$

TL 16 : $0.20x_1 + 0.13x_2 \leq 80$

Star 200 : $+1.00x_3 \geq 150$

Star 160 : $+1.00x_4 \geq 70$

Star 250 : $+1.00x_5 \geq 290$

Totally : $1.00x_1 + 1.00x_2 + 1.00x_3 + 1.00x_4 + 1.00x_5 = 750$

Obviously, all unknown variables must be nonnegative:

$x_j \geq 0, j = 1,2,3,4,5.$

In this LFP problem we have the objective function to be maximized, 6 main constraints and 5 unknown variables. Note that it would be more realistic to restrict variables x_j to integer values. Indeed, if we solve this problem we obtain the following optimal solution:

$x_1^* = 232.69,\ x_2^* = 0.00,\ x_3^* = 150.00,\ x_4^* = 70.00,\ x_5^* = 297.31$

which means that, for example the quantity of refrigerators Lebel 220 and Star 250 to be produced is 232.69 and 297.31 units respectively. Obviously, such an optimal solution cannot be applied in real life.

4.5 A Location Problem

One of the best known and most widely used discrete location models is the so-called un-capacitated facility location problem. The problem may be described as follows: there is a discrete set of possible locations for given facilities, and a set of consumers with known demands for production to be produced. The aim of optimization is to find such a location for facilities which satisfies all given constraints for demand, and maximizes the profit or the efficiency calculated as the profit/cost ratio (sometimes in the special literature referred to as a profitability index). Facilities are assumed to have unlimited capacity (un-capacitated facility), i.e. any facility can satisfy the demand of all consumers. In the case if each facility can only supply demand up to a given limit, the problem is called the capacitated facility location problem.

In its most general form, the un-capacitated facility location problem in LFP form may be formulated as follows. Let I = { 1,2,…,m} denote the set of consumers and J = { 1, 2, ... , n} the set of sites where the given facilities may be located. Let also $f_j$denote the fixed cost of opening facility in site j, and C_{ij} the profit associated with satisfying the demand of consumeri from facility j.

Usually, C_{IJ} is a function of the production costs at site j, the demand and selling price of consumer i, and the transportation costs between consumer i and site j. Obviously, without loss of generality we can assume that the fixed costs f_J are nonnegative. Introducing variables

$$y_j = \begin{Bmatrix} 1, facility\ j\ is\ open\ \ j = 1,2,\dots,n, \\ 0, otherwise \dots\dots\dots\dots\dots\dots\dots\dots\dots. \end{Bmatrix}$$

and $x_{ij} \geq 0$ 0, i = 1, 2, ... , m, j = 1, 2, ... , n, where $x_{ij}$is an unknown fraction of the demand of consumer i served by facility j, we can formulate the un-capacitated facility location problems in the following form

$$Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}}{\sum_{j=1}^{n} f_j\, y_j} \rightarrow max$$

subject to

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{j=1}^{n} f_j y_j \geq P_{min} \quad \_\_\_\_ (*)$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,2,\dots.,m,$$

$$x_{ij} \leq y_j, \ \ i = 1,2,\dots,m, \ \ j = 1,2,\dots,n,$$

$$x_{ij} \geq 0, \quad i = 1,2,\dots,m, \ \ j = 1,2,\dots,n,$$

$$y_j = 0\ or\ 1, \ \ j = 1,2,\dots,n,$$

where it is assumed that $f_i \geq 0$, j = 1, 2, ... , n, and P_{min} > 0. Additional constraint (*) here guarantees a minimum profit P_{min} .Note that the given LFP problem contains the discrete unknown variables y_j. and hence, belongs to the class of integer LFP problems

CONCLUSION

LFP Problems deals with determining the best possible allocation of available resources to meet certain specifications. In particular, they may deal with situations where a number of resources such as people, materials, machines and land are available and are to be combined to yield several products. In linear fractional programming, the goal is to determine a permissible allocation of resources that will maximize or minimize some specific showing, such as profit gained per and of cost or cost of until of product produced etc. LFP deals with that class of mathematical programming problems in which the relation must be in linear form and the function to be optimized must be a ratio of two linear functions.

In a typical maximum problem, a manufacture may wish to use available resources to produce several products. The manufacturer, knowing how much profit and cost are made for each unit of product produced, would wish to produce that particular combination of products that would maximize the profit gained per unit of cost.

Transportation problem comprise a special class of linear fractional programming. In a typical problem of this type the trucking company may be interested in finding the least expensive way of transporting each unit of large quantities of a product from a number of warehouse to a number of stores

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