STUDY ON LATTICE MATRICES

PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR

THE MASTER DEGREE IN MATHEMATICS

BY

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2018 - 2020

CERTIFICATE

This is to certify that the project entitled "**STUDY ON LATTICE MATRICES** " is a bonafide record of studies undertaken by **JOSHMI JOY (Reg no. 180011015183)**, in partial fulfillment of the requirements for the award of M.Sc. Degree in Mathematics at Department of Mathematics, St. Paul's College, Kalamassery, during 2018 – 2020.

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DECLARATION

I **JOSHMI JOY** hereby declare that the project entitled **" STUDY ON LATTICE MATRICES"** submitted to department of Mathematics St. Paul's College, Kalamassery in partial requirement for the award of M.Sc Degree in Mathematics, is a work done by me under the guidance and supervision of **Dr SAVITHA K.S ,** Department of Mathematics, St. Pauls's College , Kalamassery during 2018 – 2020.

I also declare that this project has not been previously presented for the award of any other degree, diploma, fellowship, etc.

Kalamassery JOSHMI JOY

ACKNOWLEDGEMENT

I express my heartfelt gratitude to my guide **Dr SAVITHA K.S** for providing me necessary stimulus for the preparation of this project.

I would like to acknowledge my deep sense of gratitude to **Dr. Savitha K S**, Head of the department and all other teachers of the department and classmates for their help at all stages.

I also express my sincere gratitude to **Prof. VALENTINE D’CRUZ**, Principal, St. Paul’s College, Kalamassery for the support and inspiration rendered to me in this project report.

Kalamassery **JOSHMI JOY**

STUDY ON LATTICE MATRICES

CONTENTS

INTRODUCTION

The notion of lattice matrices appeared firstly in the work, ‘Lattice matrices’ by Give’on in 1964. A matrix is called a lattice matrix if its entries belong to a distributive lattice. All Boolean matrices and fuzzy matrices are lattice matrices. Lattice matrices in various special cases become useful tools in various domains like the theory of switching nets, automata theory and the theory of finite graphs . The theory of determinant of Boolean matrices appeared firstly in the work of O. B.Sokolov . Since then, a number of researchers have studied the determinant theory for Boolean matrices and lattice matrices.

B. Poplavskii introduced the notion of minor rank of a Boolean matrix and discussed some of its properties. Further E. E. Marenich discussed the determinant rank for matrices over a Browerian, distributive lattice with 1 and 0 .The eigenproblems and characteristic roots of matrices over a complete and completely distributive lattice with the greatest element 1 and the least element 0 were also studied .

Y. J. Tan discussed the eigenproblems of lattice matrices and provided the least element for the set of all characteristic roots of a lattice matrix. Further, G. Joy and K. V. Thomas discussed the eigenproblems of nilpotent lattice matrices and introduced the concept of non-singular lattice matrices. Also, K. V.Thomas and G. Joy studied the characteristic roots of different types of lattice matrices and introduced the concept of similar lattice matrices. The least element for the set of all characteristic roots of a lattice matrix is taken as the determinant of a lattice matrix.

Lattice matrices in various special cases become useful tools in various domains like the theory of switching nets, automata theory and the theory of finite graphs .

1.PRELIMINARIES

Basic Lattice Theory

Definition 1.1 PARTIALLY ORDERED SET

A partially ordered set is an algebraic system in which a binary relation $x \leq y$ is defined, which satisfies the following postulates.

P_1 For all x, $x \leq x$. (reflexive property)

P_2 If $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetric property)∩

P_3 If $x \leq y$ and $y \leq z$ then $x \leq z$. (transitive property)

Example

- Any subset of R with usual order is a poset.
- Any collection of sets with inclusion relation is a poset.

Definition 1.2 LATTICE

A lattice L is a partially ordered set in which every pair of elements {x,y} of L have a least upper bound or join, denoted by $x \vee y$; and a greatest lower bound or meet, denoted by $x \wedge y$.

Example

- Let S={1,2,3}

  Then P(S)={Ø,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}

  (P(S), ⊆) is a poset

  Define meet and join by

  $A \wedge B = A \cap B$

  $A \vee B = A \cup B$

  Then (P(S), ⊆) is a Lattice

- Let N be the set of all positive integers

  $x \leq y$ iff x divides y

  $(N, \leq)$ is a poset

  Define meet and join by

  $x \Lambda y = GCD(x,y)$

  $x V y = LCM(x,y)$

  .Then $(N, \leq)$ is a Lattice

Remark

For x,y, and z in any lattice L, the following identities hold:

(1) $x \,V\, y = y \,V\, x$; $x \,\Lambda\, y = y \,\Lambda\, x$, (commutative laws)

(2) $(x \,V\, y) V\, z = x \,V\, (y \,V\, z)$;$(x \,\Lambda\, y) \,\Lambda\, z = x \,\Lambda\, (y \,\Lambda\, z)$, (associative laws)

(3) $x\, V(x \,\Lambda\, y) = x$; $x \,\Lambda\, (x \,V\, y) = x$, (absorptive laws)

(4) $x \,V\, x = x$; $x \,\Lambda\, x = x$. (idempotent laws)

Definition 1.3 LEAST AND GREATEST ELEMENT

If a lattice L has an element 'a' such that any element x of L satisfies the inequality $a \leq x$, then a is called the least element of L.

If a lattice L has an element 'b' such that any element x of L satisfies the inequality $b \geq x$, then b is called the greatest element of L.

We use 0 and 1 to denote the least and greatest element of L respectively.

Remark

By the definition of the ordering of lattices, the least element 0 and the greatest element 1 of the lattice L satisfy the identities:

(1) $0 \,\Lambda\, x = 0$, $0 \,V\, x = x$

(2) $1 \wedge x = x$, $1 \vee x = 1$ for all $x \in L$.

Definition 1.4 DISTRIBUTIVE LATTICE

A lattice L will be called distributive if and only if it satisfies

1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for $x,y,z \in L$.

Definition 1.5 COMPLETELY DISTRIBUTIVE LATTICE

A lattice L is said to be completely distributive if for any $x \in L$ and any family of elements $\{y_i \mid i \in I\}$, I being an index set , there are always

1) $x \wedge (\vee_{i\in I} y_i) = \vee_{i\in I}(x \wedge y_i)$

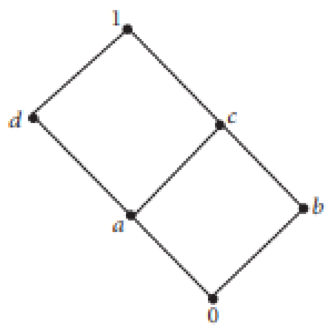
2) $x \vee (\wedge_{i\in I} y_i) = \wedge_{i\in I}(x \vee y_i)$

Definition 1.6 LATTICE MATRICES

A matrix is called a lattice matrix if its entries belongs to a distributive lattice.

EXAMPLE:

Consider the distributive lattice $L=\{0,a,b,c,d,1\}$ whose diagram is as follows



Then $A=\begin{bmatrix} a & 0 & d \\ d & a & b \\ 0 & 0 & a \end{bmatrix}$ is a Lattice Matrix

2.THE ALGEBRA OF LATTICE 0MATRICES

2.1. DEFINITIONS AND IMMEDIATE PROPERTIES

The set V,(L) of all column vectors over L forms a complete and completely distributive lattice if we make the following definitions.

$$\xi \le \eta = x_i \le y_i \text{ for all i}$$

$$\xi=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \eta=\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\xi \vee \eta = \begin{bmatrix} x_1 \vee y_1 \\ \vdots \\ x_n \vee y_n \end{bmatrix}$$

$$\xi \wedge \eta = \begin{bmatrix} x_1 \wedge y_1 \\ \vdots \\ x_n \wedge y_n \end{bmatrix}$$

Let $0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ and $e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ The vectors 0 and e are called the zero vector and the universal vector of Vn(L), respectively.

Let e_i denote the vector in Vn(L) with 1 as ith coordinate, 0 otherwise. The multiplication of the vector ξ by a scalar a in L is defined by

$$a\xi=\begin{bmatrix} a \wedge x_1 \\ \vdots \\ a \wedge x_n \end{bmatrix}$$

ξ^T denotes the row vector whose transpose is ξ and the norm of the vector ξ is defined by $\|\xi\| = \bigvee_{i=1}^{n} \xi_i$

A nonempty subset V of Vn(L) is called a vector space in Vn(L) if it is closed under "V" and under multiplication by scalars (elements of L).

Let L be a complete and completely distributive lattice with the greatest element 1 and the least element 0.The set $M_n(L)$ of all n x n matrices over L forms a completely distributive lattice if we make the following definitions:

For A=(a_{ij}) , B=(b_{ij}), C=(c_{ij}) in $M_n(L)$ define

A V B=C⟺ $a_{ij} \vee b_{ij} = c_{ij}$ for i.j=1,2,…,n

A Λ B=C⟺ $a_{ij} \wedge b_{ij} = c_{ij}$ for i.j=1,2,…,n

A ≤ B ⟺ $a_{ij} \leq b_{ij}$ for all i.j=1,2,…,n

<u>Matrix multiplication in $M_n(L)$</u>

$$AB=C \Leftrightarrow \bigvee_{k=1}^{n}(a_{ik} \wedge b_{kj})= c_{ij}$$

<u>Multiplication of matrix by a scalar</u>

$$\lambda A = B \Leftrightarrow \lambda \wedge a_{ij} = b_{ij} \text{ for i,j=1,2,…,n.}$$

<u>Premultiplication of a vector ξ by A</u>

$$A\,\xi = \eta \Leftrightarrow \bigvee_{k=1}^{n}(a_{ij} \wedge \xi_j)= \eta_i \text{ for i=1,2,…n.}$$

Let L be a distributive lattice with 0 and 1.The l.u.b, and g.l.b, of a, b belonging to L will be denoted by a + b respectively .Let L_n and a.b (or ab), +(for n > 0) be the set of n X n matrices over L.We shall use early Roman capitals as variables over L_n , and denote by $a_{ij}$the element of L which stands in the (i, j)th entry of A.

We define:

A +B = C iff $c_{ij}= a_{ij}+b_{ij}$

A ≤ B iff A +B = B, i.e., iff $a_{ij}\leq b_{ij}$

A∩B = C iff $c_{ij} = a_{ij}.\, b_{ij}$

A.B = AB = C iff $c_{ij} = \sum a_{ij}.\, b_{ij}$

A= C iff $c_{ij} = a_{ij}$

For a ϵ L aA = a.A = C iff c_{ij} = a.Aa_{ij}

(I) (i,j)=1 if i=j

(II) (i,j)=0 if i≠j

$A^{\circ} = I$, $A^{k+l} = A^k.A$,

The following special properties, most of which will be useful in the sequel, are derived immediately from these definitions:

a. The multiplication in L_n :

(1) A(BC) = (AB)C,

(2) AI = IA = A,

(3) A0 = 0A = 0,

(4) $A^p.A^q = A^{p+q}$,

(5) $(A^p)^q = A^{p.q}$

b. The multiplication and addition in L_n :

(6) A(B + C) = AB + AC,

(7) (A + B)C = AC + BC,

(8) if A ≤ B and C ≤ D then AC≤ BD,

(9) A + A = A

c. The transposition in L_n:

(10) $(A+B)^T = A^T + B^T$

(11) if A ≤ B then $A^T \leq B^T$

(12) $(A \cap B)^T = A^T \cap B^T$

(13) $(A.B)^T = B^T.A^T$

(14) $(A^T)^T = A$.

d. L_n as an algebra:

(15) L_n is a distributive lattice with zero (0) and one (E) with respect to the operations of ∩ and + ,
(16) L_n is a semigroup with the identity element I (hence, L_n is a monoid) and with zero (0) with respect to the multiplication.

2.2. IMPORTANT PROPERTY OF THE POWERS OF L_n-MATRICES

One of the most important properties of the algebra of L_n -matrices is given by the following theorem:

THEOREM 2.2.1
If S is any nonempty finite set of L_n -matrices and L_n(S) is the minimal set of L_n -matrices which includes S and is closed under multiplication and addition, then L_n (S) is finite.
L_n (S) is in fact the subalgebra of L_n generated by S.)

PROOF:

Let T = {$a_1,a_2,...a_m$ } be the set of all the elements of $L_n$which occur in the matrices of S, and let L(T) be the set of all the elements of L which are obtained by a finite number of multiplications and additions of elements of T (clearly, L (T) is the sublattice of L which is generated by T). Since L is distributive, each element of L (T) can be represented as a polynomial , i.e., as a finite sum of monomials, each monomial being a finite product of elements of T. The multiplication and addition in L are commutative and idempotent; thus, every monomial in L (T) is equal to a monomial of the form $(a_1)^{e1}$,$(a_2)^{e2}$ ...$(a_m)^{em}$ where ei (for any $1 \leq i \leq m$) is zero or one. Therefore, there are no more that 2^m unequal monomials and no more than 2^{2m} elements in L (T) (i.e., the sub lattice generated by a finite set of elements of a distributive lattice is finite; or, in other words, any distributive lattice is locally finite). Now, each element of L_n which occurs in an L_n-matrix which is in $L_n(S)$ is an element of L (T); hence, at most $(2^{(2m)})^{n2}$ different matrices can be elements of L_n (S). Anyhow, L_n (S) is finite.

2.3 ORTHOGONAL LATTICE MATRICES

In this section, a generalization and development of R. D. Luce's discussion on orthogonal Boolean matrices is given.

DEFINITION

An L_n -matrix A is called a unit iff there is an L_n -matrix B such that AB = BA = I. A is called orthogonal iff $AA^T = A^TA=I$.

LEMMA 2.3.1

(i} If CB = E then EB = E;

(ii) If EAB = E thenEB = E;

PROOF:

(i) It is always true that EB $\leq$ E and C $\leq$ E.

Therefore CB $\leq$ EB, i.e., CB = E implies E $\leq$ EB.

Thus, CB = E implies EB = E.

(ii) This is a special case of (i).

THEOREM 2.3.2.

If A is a unit then A is orthogonal.

PROOF:

If A is a unit then there is a B such that AB = BA = I and therefore $B^TA^T = A^T B^T = I$ too.

Hence, E = EAB = EBA = $EB^T A^T = EA^T B^T$, and therefore we have

$I \leq A^T A$,

$I \leq AA^T$, $I \leq B^T B$ and $I \leq BB^T$.

Thus, in order to prove that A is orthogonal, it is sufficient to show that $A^T A \leq I$ and $AA^T \leq I$ hold, i.e., to show that $A^T A \leq BA$ and $AA^T \leq AB$ hold. For this, it is suficient to show that $A^T \leq B$.

Now, since $I \leq B^T$ holds, we have $A^T \leq A^T B^T B$, but $A^T B^T = I$ and therefore $A^T \leq B$ holds and the theorem follows.

THEOREM 2.3.3.

If AB = I then A and B are roots of I.

COROLLARY :2.3.4

If A has some inverse then A is a unit.

COROLLARY :2.3.5

A is orthogonal iff A is a root of I.

COROLLARY :2.3.6

A is orthogonal iff one of the following holds:

(i) for each B there is an X such that XA = B;

(ii) for each B there is an X such that AX = B.

we can derive a structural characterization of the orthogonal L_n -matrices; and furthermore, we can establish a connection between these and the n X n permutation matrices which are the orthogonal L_n -matriees whose elements are 0 and l.

DEFINITION

a. A set $\{a_1,a_2,...a_m\}$ of elements of L is a decomposition of 1 in L iff $\sum a_r = 1$.

b. A set $\{a_1,a_2,...a_m\}$ of elements of L is orthogonal iff $a_p a_r = 0$ holds for any and provided that ν and μ such that ν≠μ. A^m

c. A set of elements of L is an orthogonal decomposition of l in L iff it is orthogonal and a decomposition of 1 in L.

d. Let A be a lattice matrix then $\{A_1, A_2, \ldots, A_n\}$ is said to be a decomposition of A if $\sum_{i=1}^{n} A_i = A$

$$A=\begin{pmatrix} 0 & a & a \\ b & b & 0 \\ 0 & 0 & a \end{pmatrix}=\begin{pmatrix} 0 & a & a \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}+\begin{pmatrix} b & 0 & 0 \\ b & b & 0 \\ 0 & 0 & 0 \end{pmatrix}= A_1+A_2$$

e. An L_n -matrix A is an orthogonal combination of matrices iff there is an orthogonal decomposition of 1 in L, $\{a_1, a_2, \ldots a_m\}$., and a set of L_n-matrices, $\{A_1, A_2, \ldots A_n\}$, such that $A = \sum a_r A_r$.

LEMMA:

A L_n-matrix is orthogonal iff each row and each column of it is an orthogonal decomposition of 1 in L.

3.CHARACTERISTIC ROOT OF LATTICE MATRICES

3.1EIGENVECTORS AND EIGENVALUES

Definition: Relative Pseudocomplement

For a, $b \in L$, the largest x satisfying the inequality $a \wedge x \leq b$ is called the *relative pseudocomplement* of a in b, and is denoted by $a \rightarrow b$.

The set V,(L) of all column vectors over L forms a complete and completely distributive lattice if we make the following definitions.

$$\xi \leq \eta = x_i \leq y_i \text{ for all i}$$

$$\xi = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \eta = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\xi \vee \eta = \begin{bmatrix} x_1 \vee y_1 \\ \vdots \\ x_n \vee y_n \end{bmatrix}$$

$$\xi \wedge \eta = \begin{bmatrix} x_1 \wedge y_1 \\ \vdots \\ x_n \wedge y_n \end{bmatrix}$$

$$\xi \rightarrow \eta = \begin{bmatrix} x_1 \rightarrow y_1 \\ \vdots \\ x_n \rightarrow y_n \end{bmatrix}$$

Let $0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ and $e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ The vectors 0 and e are called the zero vector and the universal vector of Vn(L), respectively.

Let $A \in M_n(L)$, an *eigenvector* of A is a vector $\xi \in V_n(L)$ such that,

$$\mathbf{A}\, \xi = \lambda\, \xi$$

for some scalar λ. The element λ is called the associated *eigenvalue.*

It will transpire that every element of L is an eigenvalue of every matrix A and that a given eigenvector may have a variety of eigenvalues, in the classical case only the zero vector has a range of eigenvalues and it is usual to stipulate that an eigenvector is non-zero. In the case of matrices over a lattice there seems to be no advantage in making this restriction and we shall therefore admit the possibility that an eigenvector is the zero vector.

We first consider a given eigenvector of a matrix A in $M_n(L)$ and determine the range of its eigenvalues.

Theorem 3.1.1

Let A ∈ Mn(L) and λ be a given eigenvalue of A. Then

1. The set E(A, λ) of the eigenvectors of λ forms a subspace of Vn(L) with the maximum element ξ*(λ), namely the union of all eigenvectors of λ, and the smallest element 0;
2. $\xi^*(\lambda)=(\lambda e \vee A^T e) \rightarrow (\lambda A^n e)$.

Proof. (1) Proof is trivial

(2) we have

$$\lambda(\lambda e \vee A^T e) \rightarrow (\lambda A^n e) = \lambda(((\lambda e) \rightarrow (\lambda A^n e)) \wedge ((A^T e) \rightarrow (\lambda A^n e)))$$

$$= (\lambda e \wedge ((\lambda e) \rightarrow (\lambda A^n e))) \wedge ((A^T e) \rightarrow (\lambda A^n e)))$$

$$= (\lambda e \wedge \lambda A^n e) \wedge (A^T e) \rightarrow (\lambda A^n e)$$

$$= (\lambda A^n e) \wedge (A^T e) \rightarrow (\lambda A^n e)$$

$$= \lambda A^n e$$

On the other hand, for any $i \in \{1, 2, . . . , n\}$, we have

$$(A((\lambda e \vee A^T e) \to (\lambda A^n e)))_i$$

$$= \vee_{j=1}^{n} (a_{ij} \wedge ((\lambda e \vee A^T e) \to (\lambda A^n e))_j)$$

$$= \vee_{j=1}^{n} (a_{ij} \wedge \left(\left(\lambda \vee \left(\vee_{j=1}^{n} a_{ij} \right) \right) \to (\lambda A^n e)_j \right))$$

Since $a_{ij} \leq \vee \left(\vee_{j=1}^{n} a_{ij} \right)$ we have

$$(A((\lambda e \vee A^T e) \to (\lambda A^n e)))_i = \vee_{j=1}^{n} (a_{ij} \wedge (\lambda A^n e)_j)$$

$$= (\lambda A^{n+1} e)_i = (\lambda A^n e)_i$$

Then

$$A((\lambda e \vee A^T e) \to (\lambda A^n e) = \lambda A^n e$$

Therefore $(\lambda e \vee A^T e) \to (\lambda A^n e) \in E(A, \lambda)$ and so $\xi^*(\lambda) \geq (\lambda e \vee A^T e) \to (\lambda A^n e)$.

Now let $\xi \in E(A, \lambda)$ .Then $A \xi = \lambda \xi$, and so $\lambda \xi = \lambda A^n \xi \leq \lambda A^n e$. Therefore $\xi \leq (\lambda e) \to (\lambda A^n e)$.

On the other hand, since

$$A\xi = \lambda\xi \Leftrightarrow (A\xi)_i = \lambda \wedge \xi_i \text{ for } i = 1, 2, . . . , n,$$

We have

$$a_{ij} \wedge \xi_i \leq \lambda \wedge \xi_i \leq \lambda \text{ for } i, j = 1, 2, . . . , n,$$

And so

$$\xi_j \leq a_{ij} \to \lambda \text{ for } i, j = 1, 2, . . . , n.$$

Thus, we have

$$\xi_j \leq \wedge_{i=1}^{n}(a_{ij} \to \lambda) = (\vee_{j=1}^{n} a_{ij}) \to \lambda = (A^T e)_j \to \lambda \text{ for } j = 1, 2, . . . , n,$$

and so $\xi \leq (A^T e) \to (\lambda e)$. Therefore, we have

$$\xi \leq ((ATe) \to (\lambda e)) \wedge ((\lambda e) \to (\lambda Ane)) \leq (ATe) \to (\lambda Ane)$$

And so

$$\xi \leq ((\lambda e) \to (\lambda A^n e)) \wedge ((A^T e) \to (\lambda A^n e))$$

$$= (\lambda e \vee A^T e) \to (\lambda A^n e)$$

Hence

$\xi^*(\lambda) = (\lambda e \vee A^T e) \to (\lambda A^n e)$.

This proves the theorem.

Example 3.1.3

Consider the matrix $A = \begin{bmatrix} 0.5 & 0.3 \\ 0.4 & 0.5 \end{bmatrix}$ over the fuzzy algebra [0,1] and the eigen value $\lambda = 0.5$. Then we have the maximum eigenvector $\xi^*(0.5) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ but $\lambda A^2 e = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

Definition

The subspace $E(A, \lambda)$ in Theorem 3.1 is called the λ-eigenspace of A.

The following notations are used.

$[a, b] = \{x \in L \mid a \leq x \leq b\}$ is an interval in L;

$a' = a \to 0$.

Theorem 3.1.4.

Let $A \in Mn(L)$ and $\lambda \in L$. If the only eigenvector of λ is the zero vector, then $\lambda \in [0, \|A^n\|']$.

Proof.

Suppose that the only eigenvector of λ is the zero vector. For this λ we have

$\xi^*(\lambda) = 0$. That is, $(\lambda e \vee ATe) \to (\lambda Ane) = 0$. But

$\lambda A^n e \leq (\lambda e \vee A^T e) \to (\lambda A^n e)$

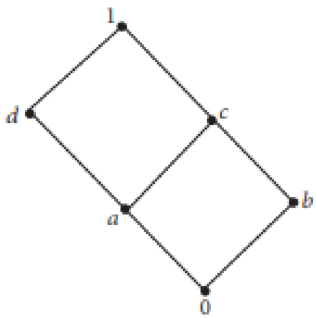
we have $\lambda A^n e = 0$, and so $A^n e \leq \lambda' e$. Therefore $\|A^n\| = e^T A^n e \leq e^T \lambda' e = \lambda'$ and so $\|A^n\|' \geq \lambda'' \geq \lambda$. This completes the proof.

Remark 3.1.5 Rutherford obtained the following result

A Boolean matrix A of order n has at most one eigenvalue whose only eigenvector is the zero vector. If such an eigenvalue exists, it is $\|A^n\|'$.

However, this result does not hold for matrices over a distributive lattice in generally.

Example 3.1.6 Consider the lattice $L = \{0, a, b, c, d, 1\}$ whose diagram is as follows.



It is easy to see that L is a distributive lattice.

Now let $A = \begin{bmatrix} a & 0 & d \\ d & a & b \\ 0 & 0 & a \end{bmatrix} \in M_3(L)$

Then $A^T e = \begin{pmatrix} 1 \\ a \\ 1 \end{pmatrix}$ and $A^3 e = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$.

For any $\lambda \in L$, we have

$$\xi^*(\lambda) = (\lambda e \vee A^T e) \to (\lambda A^3 e) = \begin{pmatrix} \lambda \vee 1 \\ \lambda \vee a \\ \lambda \vee 1 \end{pmatrix} \to \begin{pmatrix} \lambda \wedge a \\ \lambda \wedge a \\ \lambda \wedge a \end{pmatrix}$$

$$= \begin{pmatrix} 1 \to (\lambda \wedge a) \\ (\lambda \vee a) \to (\lambda \wedge a) \\ 1 \to (\lambda \wedge a) \end{pmatrix} = \begin{pmatrix} (\lambda \wedge a) \\ (\lambda \vee a) \to (\lambda \wedge a) \\ (\lambda \wedge a) \end{pmatrix}$$

Therefore

$$\xi^*(0) = \begin{pmatrix} 0 \\ a \to 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{d} \\ 0 \end{pmatrix}, \qquad \xi^*(a) = \begin{pmatrix} a \\ a \to a \\ a \end{pmatrix} = \begin{pmatrix} a \\ 1 \\ a \end{pmatrix},$$

$$\xi^*(\mathrm{b}) = \begin{pmatrix} 0 \\ c \to 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi^*(c) = \begin{pmatrix} a \\ c \to a \\ a \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \end{pmatrix},$$

$$\xi^*(\mathrm{d}) = \begin{pmatrix} 0 \\ 1 \to 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi^*(1) = \begin{pmatrix} a \\ 1 \to a \\ a \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}.$$

It is clear that *A* has two eigenvalues *b* and *d* whose only eigenvector is the zero vector.

Next, we consider a given eigenvector ξ of a matrix *A* in $M_n(L)$ and determine the range of its eigenvalues.

Theorem 3.1.7

Let A ∈ Mn(L) and ξ be an eigenvector of A. Then the eigenvalues of ξ form a sublattice of L consisting of the interval $[\lambda^0, \lambda^*]$, where

$\lambda^0 = ||A\xi||$ and $\lambda^* = ||\xi|| \to ||A\xi||$.

Corollary 3.1.8

Let ξ be an eigenvector of A. Then ξ has a unique eigenvalue if and only if

$$||A\xi|| = ||\xi|| \to ||A\xi||.$$

Theorem 3.1.9

If an eigenvector ξ of A has the unique eigenvalue λ, then every eigenvector η of λ with $\|\eta\| \geq \|\xi\|$ has a unique eigenvalue, namely λ. In particular,the maximum eigenvector $\xi^*(\lambda)$ of λ has the unique eigenvalue λ.

Proof.

we have

$\lambda = \|A\xi\|=\|\xi\|\to\|A\xi\|$.

Therefore

$\|\eta\|\to\|A\eta\|=\|\eta\|\to\|\lambda\eta\|=\|\eta\| \to (\lambda \wedge\|\eta\|)$

$= \|\eta\| \to \lambda$

$\leq\|\xi\| \to \lambda$

$= \|\xi\|\to\|A\xi\|=\|A\xi\|=\|\lambda\xi\| = \lambda \wedge \|\xi\|$

$0\leq \lambda \wedge \|\eta\|=\|\lambda\eta\|=\|A\eta\|$.

On the other hand, $\|A\eta\|\leq\|\eta\|\to\|A\eta\|$. Therefore $\|A\eta\|=\|\eta\|\to\|A\eta\|$. η has the unique eigenvalue λ. Since $\xi^*(\lambda)\geq \xi$, we have $\|\xi^*(\lambda)\| \geq \|\xi\|$, and so $\xi^*(\lambda)$ has the unique eigenvalue λ. This completes the proof.

Theorem 3.1.10.

If an eigenvector ξ of $A \in Mn(L)$ has the unique eigenvalue λ, then $0 \leq\lambda \leq\|A^n$.

Proof.

Let $\xi^*(\lambda)$ be the maximum eigenvector of λ and λ^0 the minimum eigenvalue of $\xi^*(\lambda)$. Then $\xi^*(\lambda) = (\lambda e \vee A^Te) \to (\lambda A^ne)$ and $\lambda^0 =\|A\xi^*(\lambda)\|$

Therefore,

$\lambda = \lambda^0 = \|A\xi^*(\lambda)\| = \|\lambda\xi^*(\lambda)\|$ (because $A\xi^*(\lambda) = \lambda\xi^*(\lambda)$)

$= e^T(\lambda\xi^*(\lambda)) = e^T(\lambda((\lambda e \vee A^Te) \to (\lambda A^ne)))$

$= e^T(\lambda A^ne)$

$= \lambda e^TA^ne = \lambda \| A^n \|$,

so that $\lambda\leq \| An\|$. This proves the theorem.

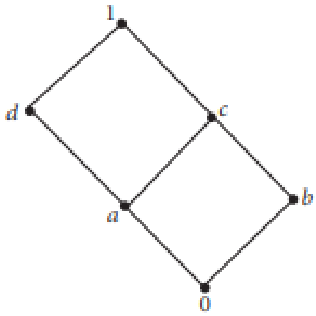
Corollary 3.1.11.

Let $A \in M_n(L)$ and λ be an eigenvalue of A. If $\lambda \in [0, \| A^n\|]$, then every eigenvector of λ has at least two eigenvalues. In particular, the maximum eigenvector $\xi^*(\lambda)$ of λ has at least two eigenvalues.

Example 3.1.12.

Consider the lattice

$L = \{0, a, b, c, d, 1\}$ whose diagram is as follows.



It is easy to see that *L* is a distributive lattice.

Now let $A = \begin{bmatrix} a & 0 & d \\ d & a & b \\ 0 & 0 & a \end{bmatrix} \in M_3(L)$

Consider the the maximum eigenvectors$\xi *(\lambda)$ for the eigenvalues λ of A. Since $\|A^3 \| = a$, we have b, c, d, $1 \in [0, \| A^3 \|]$. By the eigenvector $\xi *(\lambda)$ have at least two eigenvalues for λ = b, c, d and 1. In fact, the eigenvalues of $\xi *(b) = \xi *(d) = (0, 0, 0)^T$ are all the elements in L, and the eigenvalues of $\xi *(c) = \xi *(1) = (a, a, a)^T$ are the elements a, c and 1. For $\lambda = 0$, the eigenvector $\xi *(0) = (0,d, 0)^T$ has two eigenvalues, namely 0 and a, For $\lambda = a$, the eigenvector $\xi^*(a) = (a, 1, a)^T$ has the unique eigenvalue, namely the eigenvalue a.

We now suppose that λ is a given element in L and ξ is a given vector in $V_n(L)$ and proceed to determine the matrix A such that ξ is an eigenvector of A and λ is the associated eigenvalue. Tan obtained the maximum matrix $M(\lambda, \xi)$ in $T (\lambda, \xi)$, where $T (\lambda, \xi) = \{A \in Mn(L)|A\xi = \lambda\xi \}$, and proved that the (i, j)th element of $M(\lambda, \xi)$ is $M(\lambda, \xi)_{ij} = \xi_j \rightarrow (\lambda \wedge \xi_i)$. In the following we shall give someproperties of the matrix $M(\lambda, \xi)$.

Primitive eigenvectors.

Definition

Let $\xi \in Vn(L)$. ξ is call a *primitive vector* if all its non-zero components are identical.Now let $A \in Mn(L)$, and $\xi \in Vn(L)$ be a primitive vector. Then ξ can be written as follows:

$\xi = a\,(\vee_{i \in U} e_i)$

where $a \in L$ and $a \neq 0$, U is a nonempty subset of $N = \{1, 2, \ldots, n\}$.

If the primitive vector ξ satisfies the equation $A\xi = \lambda\xi$, then we have

$(\vee_{i \in U} a_{ij}) \wedge a = \lambda \wedge a$ for all $i \in U$ and

$(\vee_{i \in U} a_{ij}) \wedge a = 0$ for all $i \in N - U$.

3.2 CHARACTERISTIC ROOTS OF LATTICE MATRICES

Definition

Let $A \in M_n(L)$ and $\lambda \in L$. If λ satisfies $\lambda \vee b = \lambda d \vee c$,then λ is called characteristic root or characteristic value of A.

Theorem 3.2.1

Let $A \in (L)$ and S be the set of all characteristic roots of A. Then S is a sublattice of L.

Proof.

Let $\lambda, \mu \in S$. Then $\lambda \vee b = \lambda d \vee c$ and $\mu \vee b = \mu d \vee c$.

Now

$(\lambda \vee \mu) \vee b = (\lambda \vee b) \vee (\mu \vee b)$

$= (\lambda d \vee c) \vee (\mu d \vee c)$

$= (\lambda d \vee \mu d) \vee c$

$= (\lambda \vee \mu)\, d \vee c,$

$(\lambda\mu) \vee b = (\lambda \vee b)(\mu \vee b)$

$= (\lambda d \vee c)(\mu d \vee c)$

$= (\lambda d \mu d) \vee c$

$= (\lambda\mu)\, d \vee c.$

Therefore, $\lambda \vee \mu$, $\lambda\mu \in S$. Hence, the set of all characteristic roots of A is a sublattice of L.

Corollary 3.2.2

Let $A \in M_n(L)$. Then the set of all characteristic roots of A is a subset of the set of all eigenvalues of A.

In 1998, Tan proved the existence of the least element and the greatest element for the set of all characteristic roots of A by the following theorem.

Theorem 3.2.3

Let $A \in M_n(L)$. Then the set of all characteristic roots of A is

$[(b - c) \vee (c - b), d \vee c] = \{\lambda \in L \mid (b - c) \vee (c - b) \leq \lambda \leq d \vee c\}$

, where $d \vee c = e^T A^n e$.

For simplicity,

the greatest and the least characteristic roots of A are denoted by $R^u(A)$ and $r^l(A)$, respectively. Thus,

$R^u(A) = d \vee c$

$= e^T A^n e$ and

$r^l(A) = (b - c) \vee (c - b)$

Definition:Nilpotent Lattice Matrix

A square matrix (mxm)is called nilpotent if $A^m=0$

EXAMPLE

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Linear combination of nilpotent matrices are nilpotent

Theorem 3.2.4

Let $A \in M_n(L)$. Then A is nilpotent if and only if the only characteristic root of A is 0.

Proof.

First assume that A is nilpotent. Then $A^n = 0$. Therefore,

$R^u(A) = e^T A^n e$

$= e^T 0 e$

$= 0.$

Hence, the only characteristic root of A is 0.

Conversely assume that the only characteristic root of A is 0.

Then

$R^u((A) = e^T A^n e$

$= \vee_{1 \leq i, \leq n} a^{(n)}_{ij} = 0.$

Therefore

$a^{(n)}ij = 0$, for $i, j = 1, 2, \ldots, n$.

That is, $A^n = 0$.

Hence is nilpotent.

Definition:Idempotent Lattice Matrix

A square matrix (mxm)is called nilpotent if A^m=A

EXAMPLE

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Linear combination of Idempotent matrices are Idempotent

Theorem 3.2.5

Let $A \in M_n(L)$ be an idempotent matrix.Then the greatest characteristic root of A is $\bigvee_{1\le i,j\le n} a^{(n)}_{ij}$

Proof.

We have $R^u(A) = e^T A_n e$

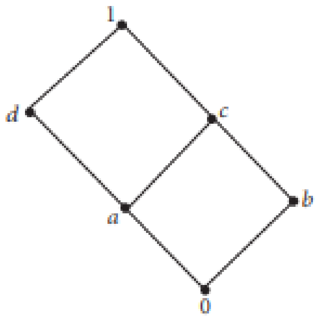
$= \bigvee_{1\le i,\le n} a^{(n)}_{ij}$ .hence,the proof is complete.

Remark

Let $A \in M_n(L)$ be idempotent. Then 0 need not be a characteristic root of A.

Example

Consider the lattice $L = \{0, a, b, c, d, 1\}$ It is easy to see that L is a distributive lattice.



Let $A = \begin{bmatrix} c & a \\ a & c \end{bmatrix} \in M_2(L)$.

Then $A^2 = \begin{bmatrix} c & a \\ a & c \end{bmatrix} = A$.

Hence, A is idempotent.

Now the characteristic roots of A are those values of λ satisfying

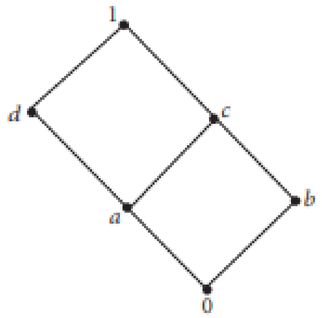
$\lambda \vee c = (\lambda \wedge c) \vee a \Rightarrow \lambda = b, c$.

Remark

Let $A, B \in M_n(L)$. Then the characteristic roots of AB and BA need not be the same.

Example

Consider the lattice $L = \{0, a, b, c, d, 1\}$ It is easy to see that L is a distributive lattice.



let A $=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

B$=\begin{bmatrix} b & c \\ d & 1 \end{bmatrix} \in M_2(L)$.

Then $AB = \begin{bmatrix} 0 & c \\ 1 & 1 \end{bmatrix}$

and BA $=\begin{bmatrix} c & c \\ c & d \end{bmatrix}$

The characteristic roots of AB are those values of λ satisfying

$\lambda \vee 0 = (\lambda \wedge 1) \vee c$

$\Rightarrow \lambda = \lambda \vee c$

$\Rightarrow \lambda = c, 1$

The characteristic roots of BA are those values of λ satisfying

$\lambda \vee a = (\lambda \wedge 1) \vee c$

$\Rightarrow \lambda \vee a = \lambda \vee c$

$\Rightarrow \lambda = b, c, 1.$

Theorem 3.2.6

Let $A \in M_n(L)$. Then the characteristic roots of A and A^T are the same.

Proof:

$R^u (A^T) = e^T (A^T)^n e$

$= e^T (A^n)^T e$

$= (e^T (A^n)^T e)^T$

$= e^T A^n e$

$= R^u(A)$,

and similarly

$r^l(A^T) = r^l(A)$.

Therefore, the characteristic roots of A and A^T are the same.

Theorem 3.2.7

Let $A \in M_{nr}(L)$. Then the set of all characteristic roots of A is L if and only if $b=c$ and $d=1$.

Proof.

First assume that the set of all characteristic roots of A is L.

Then , we have

$r^l(A) = (b-c) \vee (c-b)$

$= 0$

$\Rightarrow b - c = 0, c - b = 0$

$\Rightarrow b \leq c, c \leq b$

$\Rightarrow b = c,$

$R^u(A) = d \vee c$

$= 1$

$\Rightarrow d \vee b = 1$

$\Rightarrow d=1.$

Hence b=c and d=1

Conversely assume that b=c and d=1.Then characteristic equation may be written as

$\lambda \vee b = \lambda d \vee c$

$\Rightarrow \lambda \vee b = \lambda \vee b,$

which is true for all $\lambda \in L$.

Hence, the set of all characteristic roots of A is L.

Theorem 3.2.8

Let $A \in M_n(L)$. Then 0 is a characteristic root of A if and only if $b=c$.

Proof.

First assume that 0 is a characteristic root of A. Then,

$0 \vee b = 0d \vee c$

$\Rightarrow b = c$.

Conversely assume that $b=c$. Then ,

$\lambda \vee b = \lambda d \vee b$

$= (\lambda \vee b)\ (d \vee b)$

$= (\lambda \vee b)\ d$

$\Rightarrow (\lambda \vee b) \leq d$,

which is true when $\lambda=0$.

Hence, 0 is a characteristic root of A.

Definition

Let A, $B \in M_n(L)$. If there exists an invertible matrix $P \in (L)$ such that $B=P^{-1}AP$, then B is said to be similar to A.

Let A, $B \in M_n(L)$. If there exists an orthogonal matrix $Q \in (L)$ such that $B=Q^{-1}AQ$, then B is said to be orthogonally similar to A.

Note that, If B is similar to A, then A is similar to B. B being similar to A is the same as B being orthogonally similar to A.

4. DETERMINANT AND RANK OF A LATTICE MATRIX

4.1 THE RANK OF A LATTICE MATRIX

In this section, we introduce the concept of rank of a lattice matrix and discuss some of its properties. For any we use the notation $a > b$ to denote $b \leq a$ and $b \neq a$

Definition: Permanent of a matrix

For A∈$M_n(L)$. , the *permanent* |A| of A is defined as

$$|A| = \bigvee_{\sigma \epsilon S_n} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

S_n denotes the symmetric group of all permutations of the indices{1,2,…,n }

Let us use the following notations

$$S_n^+ = \{\sigma \epsilon S_n | \sigma \text{ is even}\}$$

$$S_n^- = \{\sigma \epsilon S_n | \sigma \text{ is odd}\}$$

Now the *semi-permanants* of A∈$M_n(L)$ are defined as follows:

$$p_n(A) = \bigvee_{\sigma \epsilon S_n^+} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

$$q_n(A) = \bigvee_{\sigma \epsilon S_n^-} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

Let A ∈ $M_n(L)$.

Then the *determinant* of A is defined

$$(p_n(A) - q_n(A)) \vee (q_n(A) - p_n(A))$$

and is denoted by D(A) .

Definition

Let $A \in M_n$ (L). The permanent rank of a nonzero matrix A is the greatest integer k for which there exists a $-(k \times k)$submatrix B of A such that $|B| > 0$. The permanent rank of is denoted by rank $_P$ (A). The permanent rank of zero matrix is 0.

The determinant rank (or rank) of a nonzero matrix is the greatest integer k for which there exists a $(k \times k)$-submatrix B of A such that D(B) >0. The rank of A is denoted by rank(A) . The rank of a zero matrix is 0 .

Theorem 4.1.1

Let A,B $\in M_n$ (L). Then

a) rank(A)$\leq$ rank$_P$(A)
b) rank(A)$\leq$ rank$_F$(A)
c) $\text{rank(AB)} \leq \min\{\text{rank(A)}, \text{rank(B)}\}$

Proof

(a) We have D(A)$\leq$| A| . Hence rank(A)$\leq$ rank $_p$ (A) .

(b) Let rank(A) = k . Then there exists a $(k \times k)$-submatrix B of A such that D(B) > 0 . By Corollary 4.6, we get rank(A) = k = rank(B) = rank $_f$ (B) $\leq$ rank (A) .

Hence rank(A)$\leq$ rank$_F$(A)

4.2 THE COLUMN RANK, ROW RANK AND FACTOR RANK OF A LATTICE MATRIX

In this section, we generalize the concepts and propositions for Boolean matrices in [8] to the case of lattice matrices.

Let be a distributive lattice with 1 and 0, and be the set of all column vectors (n -vectors) over L . Denote $\vec{e}$=(1,1, 0…,1)[7] and $\vec{0}$=(0,0,…,0)$^T \in V_n$(L).
We endow V_n(L) with the following operations:
For $\vec{x} = (x_1, x_2, \dots, x_n)^T$, $\vec{y} = (y_1, y_2, \dots, y_n)^T \epsilon V_n(L)$ and $a \epsilon L$, define
Addition: $\vec{x} \vee \vec{y} = (x_1 \vee y_1, x_2 \vee y_2, \dots, x_n \vee y_n)^T$
Scalar multiplication: $a\vec{x} = (ax_1, ax_2, \dots, ax_n)^T$
Then V_n(L) has the properties of a linear space except for the lack of additive inverse for non-zero elements. We call the elements of V_n(L) as *lattice vectors*(or L simply -*vectors*) and the elements of L as *scalars*.)
In this paper, a space means an L -space and a vector means an L -vector.

Lemma 4.2.1
For $\vec{x} \epsilon V_n(L)$ and $a \epsilon L$, we have
(a) $0\vec{x} = \vec{0}$
(b) $a\vec{0} = \vec{0}$

Let *S* be a non-empty subset of $V_n(L)$. Then *S* is a *subspace* of $V_n(L)$ if it is closed under addition and scalar multiplication.

A vector is a *linear combination* of vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$, when there exists scalars such that $\vec{y} = \bigvee_{1 \le i \le k} a_i \vec{x}_i$.

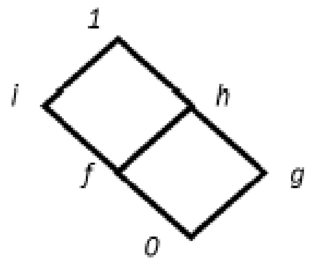
Let $S = S(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k)$be the set of all possible linear combinations of the vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$. Then S is a subspace of $V_n(L)$ and is called the *linear span* of the vectors . Here $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ is a *spanning subset* of S. Among the spanning subsets of S, the one with the smallest cardinality, d , are *bases* of S ; The *dimension* of S is that number d . The dimension of the zero space is zero.

Let $A \in M_{m,n}(L)$. Then the *column space* A of is the linear span of the set of all columns of A and is denoted by $C(A)$. The *row space* of is the linear span of the set of all rows of A and is denoted by (A) . The dimension of $C(A)$ is called the *column rank* of and is denoted by $rank_c(A)$. The dimension of $R(A)$ is called the *row rank* of and is denoted by $rank_R(A)$.

It is known that $rank_c(A) \neq rank_R(A)$, in general case. This is demonstrated in the following example.

Example

Consider the lattice L={0,f,g,h,i,1} , whose diagrammatical representation is as follows:



It is easy to see that is a distributive lattice with 0 and 1 .

Let A= $\begin{bmatrix} 0 & f & g \\ f & h & h \\ i & h & h \end{bmatrix} \in M_3(L)$

Then the column space $C(A)$ *is spanned by the vectors* $\{(0,f,i)^T, f,h,h^T, (g,h,h)^T\}$ *and this is a minimal spanning subset of* $C(A)$. *Therefore,* $rank_c(A) = 3$

Now the row space is spanned by the vectors $\{(0,f,i)^T, f,h,h^T, (g,h,h)^T\}$. *But, this is not a minimal spanning subset of . We can see that*

$(0, f, i)^T, f, h, h^T, (g, h, h)^T\}$ *is a minimal spanning subset of* $R(A)$. *Therefore,* $rank_R(A)$ *=2.*

Let $\vec{A}_j$ be the j^th column of $M_{m,n}(L)$. Then the matrix A can be written as

$A = [\vec{A}_1 \dots \vec{A}_j \dots \vec{A}_n]$.

Theorem 4.2.2

(a). Let $A \in M_{m,n}(L)$ *and* $B \in M_{m,n}(L)$ *. Then C(A)=C(B) if and only if A=BU and B=AV, for some* $U \in M_{k,n}(L), V \in M_{k,n}(L)$.

(b). Let $A \in M_{m,n}(L)$ *and* $B \in M_{m,n}(L)$ *. Then R(A)=R(B) if and only if A=UB and B=VA, for some* $U \in M_{k,n}(L), V \in M_{k,n}(L)$.

Proof.

(a) First assume that C(A)=C(B)

Then $\vec{A}_j \in C(B), 1 \leq j \leq n$. Therefore, $\vec{A}_j = \bigvee_{1 \leq l \leq k} u_{lj} \vec{B}_j \;\; 1 \leq j \leq n.$

Let U=(u_{lj}), $1 \leq l \leq k 1 \leq j \leq n$ Thus we get *A=BU* where $U \in M_{k,n}(L)$

Also, $\vec{B}_l \in C(A), 1 \leq l \leq k$. Therefore, $\vec{B}_l = \bigvee_{1 \leq l \leq k} v_{jl} \vec{A}_j \;\; 1 \leq l \leq k.$

Let V=(v_{lj}), $1 \leq j \leq n, 1 \leq l \leq k$ Thus we get *B=VA* where $V \in M_{n,k}(L)$

Conversely assume that A=BU and B=AV, for some $U \in M_{k,n}(L), V \in M_{n,k}(L)$.

Then for $1 \leq j \leq n$, , $\vec{A}_j = \bigvee_{1 \leq l \leq k} u_{lj} \vec{B}_j$ and for $1 \leq l \leq k \vec{B}_l = \bigvee_{1 \leq l \leq k} v_{jl} \vec{A}_j$.Therefore, $\vec{A}_j \in C(B), 1 \leq j \leq n$ and $\vec{B}_l \in C(A), 1 \leq l \leq k$.Thus C(A) ⊆ C(B) and C(B) ⊆ $C(A)$ Hence C(A) = C(B)

Similarly we can prove (b)

4.3 THE DETERMINANT OF A LATTICE MATRIX

Let L be a dually Brouwerian, distributive lattice with the greatest element 1 and the least element 0 , respectively. In this section, we introduce the determinant of a matrix over L , using the semi-permanants.

Definition

Let A ∈ V_n(L).Then the determinant of A is defined

$(p_n(A) - q_n(A)) \vee (q_n(A) - p_n(A))$ and is denoted by D(A) .

We have

$$\begin{aligned} D(A) &= (p_n(A) - q_n(A)) \vee (q_n(A) - p_n(A)) \\ &= (p_n(A) \vee q_n(A)) - (q_n(A) \wedge p_n(A)) \text{ (by Lemma 2.1(f))} \\ &\leq p_n(A) \vee q_n(A) \text{ (by Lemma 2.1(b)).} \end{aligned}$$

Therefore, D(A) ≤| A| .

If L is a Boolean lattice, then D(A) = $(p_n(A) \wedge \bar{q}_n(A)) \vee (q_n(A) \wedge \bar{p}_n(A))$

Now we discuss some properties of the determinant of a lattice matrix.

Theorem 4.3.1

Let A∈ M_n (L) . Then D(A) = 0 if and only if ($p_n(A) = q_n(A)$.

Proof.

We have

D(A) = 0 iff $p_n(A) - q_n(A)) \vee (q_n(A) - p_n(A))$ = 0

iff $p_n(A) - q_n(A)$ = 0, $q_n(A) - p_n(A)$= 0

iff $p_n(A) \leq q_n(A)$, $q_n(A) \leq p_n(A)$ (by Lemma 2.1(c)) iff $p_n\ (A) = q_n(A)$

Theorem 4.3.2

Let A∈ M_n (L) and λ ∈ L . Then

(a). D(A) = D(A^T)

(b). if B is the matrix obtained from A by interchanging $\bar{A}_i$ and $\bar{A}_j$, then

D(B) = D(A)

(c). if B is the matrix obtained from A by replacing $\bar{A}_j$ by $\lambda\bar{A}_j$, then D(B) ≤ λ D(A)

(d). if two columns of A are identical, then . D(A) = 0

Proof.

(a) We have

D(A^T) = $(p_n\ (A^T) - q_n(A^T))$V $(q_n(A^T) - p_n\ (A^T))$

=($p_n\ (A) - q_n(A)$)V $(q_n(A) - p_n\ (A))$ (by Lemma 2.4(b))

= D(A).

(b) We have

D(B) = $(p_n\ (B) - q_n(B))$V $(q_n(B) - p_n\ (B))$

= $(q_n(A) - p_n\ (A))$V $(p_n\ (A) - q_n(A))$ (by Lemma 2.4(c))

= D(A).

(c) We have

D(B) = $(p_n\ (B) - q_n(B))$V $(q_n(B) - p_n\ (B))$

=$(\lambda p_n\ (A) - \lambda q_n(A))$V $(\lambda q_n(A) - \lambda p_n\ (A))$

≤ $\lambda(p_n\ (A) - q_n(A))$Vλ $(q_n(A) - p_n\ (A))$ (by Lemma 2.4(d))

≤ λ ($(p_n\ (A) - q_n(A))$V $(q_n(A) - p_n\ (A))$)

≤ λ D(A).

(d) We have $p_n\ (A) = q_n(A)$. Therefore, D(A) = 0 .

Theorem 4.3.3

Let A∈ M_n (L) and $\vec{B}_i$∈V_n(L) , 1≤ l≤ k . Suppose that each column of the matrix A is a linear combination of $\vec{B}_i$ 1≤ l≤k . Then D(A) = 0 , provided that k < n .

Proof.

Suppose that $\vec{A}_J = \bigvee_{1\le l\le k} u_i\, \vec{B}_i$, for some 1≤ j≤ n . Then

$$p_n\ (A) = p_n\left([\vec{A}_1 \dots \vec{A}_J \dots \vec{A}_n]\right)$$
$$= p_n\left([\vec{A}_1 \dots \bigvee_{1\le l\le k} u_i\, \vec{B}_i \dots \vec{A}_n]\right)$$
$$= \bigvee_{1\le l\le k} u_i\, p_n\left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right)$$

Similarly

$$q_n\ (A) = \bigvee_{1\le l\le k} u_i\, q_n\left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right)$$

Therefore

$$p_n\ (A) - q_n\ (A) = \bigvee_{1\le l\le k} u_i\, p_n\left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right) - \bigvee_{1\le l\le k} u_i\, q_n\ \left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right)$$
$$\le \bigvee_{1\le l\le k}\left(u_i\, p_n\left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right) - u_i q_n\left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right)\right)$$
$$\le \bigvee_{1\le l\le k} u_i\left(p_n\left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right) - q_n\left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right)\right)$$
$$\le \bigvee_{1\le l\le k} u_i\ \mathrm{D}\left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right)$$

similarly $q_n\ (A) - p_n\ (A) \le \bigvee_{1\le l\le k} u_i\ \mathrm{D}\left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right)$

Therefore D(A) ≤ $\bigvee_{1\le l\le k} u_i\ \mathrm{D}\left([\vec{A}_1 \dots \vec{B}_i \dots \vec{A}_n]\right)$

Since this chain can be continued with the use of analogous reasoning applied to each column of the matrix A , we find that D(A) is less than or equal to some linear combination of the determinants of the square matrices constructed from {B , 1≤ l≤ k }Since k < n , the determinants of such matrices contain identical columns and therefore are equal to zero. Hence, D(A) = 0 .

Theorem 4.3.4

Let A∈ M_n (L) . If rank $_f$ (A) < n , then D(A) = 0 .

Proof.

Let rank $_f$ (A) = r < n. Then there exists $\vec{B_i}$∈V (L) , 1≤ l≤ r such that each column of the matrix A is a linear combination of B , 1≤ l≤ r. Hence from Theorem 4.4, it follows that D(A) = 0 .

Corollary 4.3.5

Let A∈ M_n (L). If D(A) > 0 , then rank_f (A) = rank_c (A) = rank $_R$(A) = n . .

Theorem 4.3.6

Let A, B∈ M_n (L). Then D(AB) ≤ D(A) ∧ D(B) .

Proof.

We have D(AB) = $(p_n\ (AB) - q_n(AB))$V $(q_n(AB) - p_n\ (AB))$

Consider$(p_n\ (AB) - q_n(AB)$

$=\bigvee_{\sigma\in S_n^+}(AB)_{1\sigma(1)}(AB)_{2\sigma(2)} \dots (AB)_{n\sigma(n)} -$

$\bigvee_{\sigma\in S_n^+}(AB)_{1\sigma(1)}(AB)_{2\sigma(2)} \dots (AB)_{n\sigma(n)}$

$=\bigvee_{\sigma\in S_n^+}(\bigvee_{1\le k_1\le k_2 \dots\le n} a_{1k_1}a_{2k_2} \dots a_{nk_n}b_{k_1\sigma(1)}b_{k_2\sigma(2)} \dots b_{k_n\sigma(n)}) -$

$\bigvee_{\sigma\in S_n^-}(\bigvee_{1\le k_1\le k_2 \dots\le n} a_{1k_1}a_{2k_2} \dots a_{nk_n}b_{k_1\sigma(1)}b_{k_2\sigma(2)} \dots b_{k_n\sigma(n)})$

$\le\bigvee_{1\le k_1\le k_2 \dots\le n} a_{1k_1}a_{2k_2} \dots a_{nk_n}(\bigvee_{\sigma\in S_n^+}(b_{k_1\sigma(1)}b_{k_2\sigma(2)} \dots b_{k_n\sigma(n)}) -$

$\vee_{\sigma\in S_n^-}(b_{k_1\sigma(1)}b_{k_2\sigma(2)} \dots b_{k_n\sigma(n)})$

$\leq \vee_{\tau\epsilon S_n} a_{1\tau(1)}\, a_{2\tau(2)} \dots a_{n\tau(n)}(\vee_{\sigma\in S_n^+}(b_{\tau(1)\sigma(1)}b_{\tau(2)\sigma(2)} \dots b_{\tau(n)\sigma(n)}) - \vee_{\sigma\in S_n^-}(b_{\tau(1)\sigma(1)}b_{\tau(2)\sigma(2)} \dots b_{\tau(n)\sigma(n)}))$

$\leq \quad \vee_{\tau\epsilon S_n^+} a_{1\tau(1)}\, a_{2\tau(2)} \dots a_{n\tau(n)}(\vee_{\sigma\in S_n^+}(b_{\tau(1)\sigma(1)}b_{\tau(2)\sigma(2)} \dots b_{\tau(n)\sigma(n)}) - \vee_{\sigma\in S_n^-}(b_{\tau(1)\sigma(1)}b_{\tau(2)\sigma(2)} \dots b_{\tau(n)\sigma(n)}))$
$\vee (\vee_{\tau\epsilon S_n^-} a_{1\tau(1)}\, a_{2\tau(2)} \dots a_{n\tau(n)}(\vee_{\sigma\in S_n^-}(b_{\tau(1)\sigma(1)}b_{\tau(2)\sigma(2)} \dots b_{\tau(n)\sigma(n)}) - \vee_{\sigma\in S_n^+}(b_{\tau(1)\sigma(1)}b_{\tau(2)\sigma(2)} \dots b_{\tau(n)\sigma(n)}))$

$\leq \ (\vee_{\sigma\epsilon S_n^+} a_{1\sigma(1)}\, a_{2\sigma(2)} \dots a_{n\sigma(n)}(\vee_{\sigma\in S_n^+}(b_{1\sigma(1)}b_{2\sigma(2)} \dots b_{n\sigma(n)} - \vee_{\sigma\in S_n^-}(b_{1\sigma(1)}b_{2\sigma(2)} \dots b_{n\sigma(n)}))$
$\vee (\vee_{\sigma\epsilon S_n^-} a_{1\sigma(1)}\, a_{2\sigma(2)} \dots a_{n\sigma(n)}(\vee_{\sigma\in S_n^-}(b_{1\sigma(1)}b_{2\sigma(2)} \dots b_{n\sigma(n)} - \vee_{\sigma\in S_n^+}(b_{1\sigma(1)}b_{2\sigma(2)} \dots b_{n\sigma(n)}))$

$\leq p_n\,(A)(p_n\,(B) - q_n(B)) \vee q_n(A)(q_n(B) - p_n\,(B))$.

Therefore,

$p_n\,(AB) - q_n(AB) \leq p_n\,(A)(p_n\,(B) - q_n(B)) \vee q_n(A)(q_n(B) - p_n\,(B))$.

Similarly, we can prove that
$p_n\,(AB) - q_n(AB) \leq p_n\,(B)(p_n\,(A) - q_n(A)) \vee q_n(B)(q_n(A) - p_n\,(A))$.
$q_n(AB) - p_n\,(AB) \leq p_n\,(A)(q_n\,(B) - p_n(B)) \vee q_n(A)(p_n(B) - q_n\,(B))$
$q_n(AB) - p_n\,(AB) \leq p_n\,(B)(q_n\,(A) - p_n(A)) \vee q_n(B)(p_n(A) - q_n\,(A))$

Hence,
D(AB)=($p_n\,(AB) - q_n(AB)) \vee (q_n(AB) - p_n\,(AB))$

$\leq p_n\ (A)(p_n\ (B) - q_n(B)) \vee q_n(A)(q_n(B) - p_n\ (B))) \vee (p_n\ (A)(q_n\ (B) - p_n(B)) \vee q_n(A)(p_n(B) - q_n\ (B)))$

$\leq p_n\ (A)((p_n\ (B) - q_n(B)) \vee (q_n(B) - p_n\ (B))) \vee q_n(A)((p_n(B) - q_n\ (B)) \vee ((q_n\ (B) - p_n(B)))$

$\leq (p_n\ (A) \vee q_n(A) \wedge \left(\left(p_n\ (B) - q_n(B)\right) \vee \left(q_n(B) - p_n\ (B)\right)\right) = |A| \wedge \mathrm{D(B)}$

Also

D(AB)$\leq |B| \wedge$ D(A)

Therefore,

D(AB)$\leq$ (|A|∧D(B)) ∧(|B|∧ D(A))

$\leq$(|A|∧D(A)) ∧(|B| ∧ D(B))

$\leq$ D(A) ∧ D(B)

Hence, the proof is complete.

Theorem 4.3.7

Let A∈ M_n (L) be nilpotent. Then D(A) = 0 .

Proof.

Assume that A is nilpotent. we have p (A) = q (A) = 0

Hence D(A) = 0 .

Theorem 4.3.8

Let A∈ M_n (L) be triangular (upper or lower). Then D(A) $=\bigwedge_{1\leq i\leq n} a_{ii}$.

Proof

Assume that A is upper triangular. Then by Lemma 2.5(a), we have

$p_n\ (A) = \wedge_{1\le i\le n}\, a_{ii}$ and $q_n\ (A) = 0$ Hence D(A) $=\wedge_{1\le i\le n}\, a_{ii}$

If A is lower triangular, then A is upper triangular. Hence by Theorem 4.3, we get D(A) = D(A^T) $=\wedge_{1\le i\le n}\, a_{ii}$

Theorem 4.3.9

Let A∈ M_n (L)be invertible. Then D(A) =1.

Proof.

Assume that A is invertible. Then by Lemma 2.5(b), we have

$p_n\ (A)\ \wedge q_n(A)$= 0 and $p_n\ (A)$ V $q_n(A)$ =1. Hence

D(A) = $(p_n\ (A) - q_n(A))$V$(q_n(A) - p_n\ (A))$

=$(p_n\ (A)$V $q_n(A)) - (q_n(A) \wedge p_n\ (A))$ (by Lemma 2.1 (f)

=1

Theorem 4.3.10

Let A, B∈ M_n (L) be similar lattice matrices. Then D(A) = D(B) .

Proof.

Let A, B∈ M_n (L) be similar lattice matrices. Then by Lemma 2.5(c), we have

$p_n\ (A)$ = $p_n\ (B)$ and $q_n\ (A)$ = $q_n\ (B)$. Hence D(A) = D(B) .

CONCLUSION

Since 1964, in which the notation of Lattice matrices appeared firstly , a number of researchers have studied lattice matrices .The eigenproblems and characteristic roots of matrices over a complete and completely distributive lattice opened a new door for young researchers. In this project, algebra of Lattice matrices is discussed wherein some basic properties of lattice matrices are obtained.Also this project deals with the study of characteristic root of lattice matrices and the determinant theory of Lattice matrices .It is also noted that lattice matrices in various special cases become useful tools in various domains like the theory of switching nets,automata theory and the theory of finite graphs.

REFERENCES

[1] Y.Give´on, “Latticematrices,” *Information and Control*, vol. 7, no.4, pp. 477–484, 1964.

[2] K. L. Zhang, “Determinant theory for D01-lattice matrices,”*Fuzzy Sets and Systems*, vol. 62, no. 3, pp. 347–353, 1994.

[3] Y. Tan, “On the eigenproblem of matrices over distributive lattices,” *Linear Algebra and Its Applications*, vol. 374, pp. 87–106,2003.

[4] Y.-J. Tan, “Eigenvalues and eigenvectors for matrices over distributive lattices,” *Linear Algebra and Its Applications*, vol. 283, no. 1–3, pp. 257–272, 1998.

[5] G. Birkhoff, *Lattice Theory*, American Mathematical Society, Providence, RI, USA, 3rd edition, 1967.

[6] K. V. Thomas and G. Joy, *A Study on Characteristic Roots of Lattice Matrices, Journal of Mathematics*, vol. 2016, Article ID 3964351, 8 pages, 2016. doi:10.1155/2016/3964351.