CHAOS THEORY

Project Report submitted

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MASTER OF SCIENCE IN MATHEMATICS

By

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CERTIFICATE

This is to certificate that the project entitled “ CHAOS THEORY is a bonafide record of the studies undertaken by BINTA JESSY JOSE (Reg no: 180011015182), in partial fulfillments of the requirements for the award of M.Sc. Degree in mathematics at Department of Mathematics, St.Paul’s College, Kalamassery, during 2018-2020.

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DECLARATION

I BINTA JESSY JOSE declare that, this project titled "CHAOS THEORY" has been prepared by me under the supervision of Mr. ARAVIND KRISHNAN.R , Department of Mathematics, St.Paul's College, Kalamassery.

I also declare that this project has not been submitted by me fully or partially for the awards of any degree, diploma, title or recognition earlier.

Date: ***BINTA JESSY JOSE***

Place: Kalamassery

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BINTA JESSY JOSE

CHAOS THEORY

CONTENTS

INTRODUCTION

Dynamics is a time evolutionary process. Long term predictions of some systems often become impossible. Even this trajectories cannot be represented by usual geometry. It may happen that small differences in the initial produce very great ones in the final phenomena.

Dynamical system is a branch of mathematics in which a function describes the time dependence of a point in a geometrical space. Such process occurring all branches of science. At any given time, a dynamical system has a state given by a triple of real numbers that can be represented by a appropriate state space. The evolution rule if dynamical system is a function that describes what future states follow from the current state. Often the function is a deterministic that is, for a given time interval only one future state follows from the current state. For example, the motion of the stars and the galaxies in the heaven , the stock market . The changes chemicals undergo, the rise and fall of population and the motion of simple pendulum, the flow of water in a pipe are classical examples of dynamical systems in chemistry , biology and physics .Some dynamical systems are predictable, whereas others are not . We know that sun will rise tomorrow and when you add cream to a cup of coffee, the resulting chemical reaction will not be an explosion. On the other hand predicting the weather a month from now seems impossible. We might think that the reason for this unpredictability is that there are simply too many variations present in the meteorological or economic system. That is indeed true in these cases, but this is by no means the complete answer. One of the remarkable discoveries of the twentieth century mathematics is that very simple systems, even systems depending on only one variable, may behave just as unpredictably as the stock market, just as wildly as a turbulent waterfall and just as violently as a hurricane. The culprit, the reason for this unpredictable behaviour has been called ”chaos”

Theory of chaos

Chaos is a part of mathematics . It looks at certain systems that are very sensitive and is a state of utter confusion and disorder . It teaches us to expect the unexpected . Chaotic motion are unpredictable . A very small change may make the system behave completely differently . The simple looking phenomenon such as the smoke column rising in still air from cigarette , the oscillations and their layers in the smoke column and so complicated to defy understanding.

Thus Chaos theory is the branch of mathematics focussing on the be- haviour of dynamical systems that are highly sensitive to initial condition. Let us first have an idea about what we mean by chaos in the mathematical sense.

Suppose we start with a process or an equation with a certain number and end up with a final number. The number we start with is called is the initial condition and the number we end with is called the result. Suppose we start with 1 and end up with 10. Let us change the initial condition slightly and start with 1.1 instead of 1. If we go through the same process mathe- matically, suppose we get a result close to 10 eg, 10.3 or 10.5. In this case everything would be normal and predictable. Obviously there is no chaos. But suppose the result obtained is 12 or 15 or even 20, in this case the result is totally different from 10 and there is a huge difference. Changing the initial condition slightly from 1 to 1.1, results in totally different answers. Clearly, this is not predictable. This is exactly what we mean by chaos. In this case we say that the process is chaotic.

CHAPTER 1

HISTORY OF CHAOS THEORY

Edward Lorenz was an early pioneer of the theory. His interest in chaos came about accidentally through his work on weather prediction in 1961.Lorenz was using a simple digital computer, a Royal McBee LGP-30, to run his weather simulation. He wanted to see a sequence of data again, and to save time he started the simulation in the middle of its course. He did this by entering a printout of the data that corresponded to conditions in the middle of the original simulation. To his surprise, the weather the machine began to predict was completely different from the previous calculation. Lorenz tracked this down to the computer printout. The computer worked with 6-digit precision, but the printout rounded variables off to a 3-digit number, so a value like 0.506127 printed as 0.506. This difference is tiny, and the consensus at the time would have been that it should have no practical effect. However, Lorenz discovered that small changes in initial conditions produced large changes in long-term outcome.

PRELIMINARY DEFINITIONS.

We know that there are many kinds of problems in science and mathematics that involve iteration. Iteration means to repeat a process over and over. In dynamics, the process that is repeated is the application of a function. To iterate a function means to evaluate the function over and over, using the output of the previous application as the input for the next. Mathematically, this is the process of repeatedly composing the function with itself. For example let F be a function. Then $F^2(x)$ is the second iterative of F namely F(F(x)), $F^3(x)$ is the third iterative of F i.e ,F(F(F(x))) and in general F^n (x) is the n-fold composition of F with itself.

Suppose let $F(x) = x^2 + 1$, then

$$F^2(x) = (x^2 + 1)^2 + 1$$

$$F^3(x) = ((x^2 + 1)^2 + 1)^2 + 1$$

It is important to realise that $F^n(x)$ does not mean raise F(x) to the n^{th} power, rather $F^n(x)$ is the n^{th} iterate of F evaluated at x.

ORBITS

Given $x_0 \in R$, we define the orbit of x_0 under F to be the sequence of points

$x_0, x_1 = F(x_0), x_2 = F^2(x_0), \ldots , x_n = F^n(x_0), \ldots$

The point x_0 is called the seed of the orbit.

For example :

1. If $F(x) = \sqrt{x}$ and $x_0 = 256$, the first few points on the orbit of x_0 are

$$x_0 = 256$$

$x_1 = \sqrt{256} = 16$

$x_2 = \sqrt{16} \quad = 4$

$x_3 = \sqrt{4} \quad = 2$

$x_4 = \sqrt{2} \quad = 1.414...$

2. If $S(x) = \sin(x)$, the orbits of $x_0 = 123$ is

$x_0 = 123$

$x_1 = -0.4599$

$x_2 = -0.4439$

.

.

.

.

$x_{300} = -0.0975$

$x_{301} = -0.0974$

.

.

.

(Note that here x is given in radians, not in degrees.)

Slowly the points on this orbit tends to 0.

3. If $C(x) = \cos(x)$, then the orbit of $x_0 = 123$ is

$x_0 = 123$

$x_1 = -0.8879$

$x_2 = 0.6309$

.

.

.

.

$x_{50} = 0.739085$

$x_{51} = 0.739085$

$x_{52} = 0.739085$

.

.

.

After a few iterations, this orbit seems to stop at 0.739085.

TYPES OF ORBITS

FIXED POINT ORBIT

A fixed point is a point x_0 that satisfies $F(x_0) = x_0$. Also $F^2(x_0) = F(F(x_0))= F(x_0) = x_0$. So in general, $F^n(x_0) = x_0$. So the orbit of a fixed point is the constant sequence x_0, x_0, x_0,. A fixed point never moves. As its name implies, it is fixed by the function. For example 0, 1 and -1 are all fixed points for $F(x) = x^3$, while only 0 and 1 are fixed points for $F(x) = x^2$.Fixed points are found by solving the equation $F(x) = x$. Thus $F(x) = x^2 - x - 4$ has fixed points at the solutions of $x^2 - x - 4 = x$

$=\Rightarrow x^2 - 2x - 4 = 0$

$=\Rightarrow x = 2\pm\sqrt{4} + 16/2$

$=\Rightarrow x = 2\pm\sqrt{20}/2$

$=\Rightarrow x = 2\pm2\sqrt{5}/2$

$=\Rightarrow x = 1\pm\sqrt{5}$

Fixed points may also be found geometrically by examining the intersection of the graph with the diagonal line $y = x$. For example, the following figure shows that the only fixed points of $S(x) = \sin(x)$ is at $x_0 = 0$, since that is the only point of intersection of the graph of S with the diagonal $y = x$.

Similarly $C(x) = \cos(x)$ has a fixed point at 0.739085 as shown here.

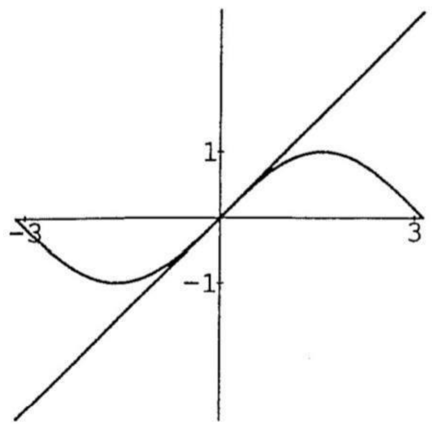

Figure 1: The fixed point of $S(x) = \sin x$ is 0.

There are two markedly different types of fixed points, attracting and repelling fixed points. Consider $F(x) = x^2$, it has two fixed points 0 and 1. If we choose any x_0 with $|x_0| < 1$, then the orbits of x_0 rapidly approaches zero. For example, the orbit of 0.1 is 0.1, 0.01, 0.0001, 0.00000001, ...

In fact any x_0 with $0 \leq x_0 < 1$, no matter how close to 1, leads to an orbit that tends far from 1 close to 0. For example the orbit of 0.9 is

0.9, 0.81, 0.6561, 0.430467, ..., 0.00117, ...

More precisely if $0 \leq x_0 < 1$, then $F^n(x0) \rightarrow 0$ as $n \rightarrow \infty$. On the other

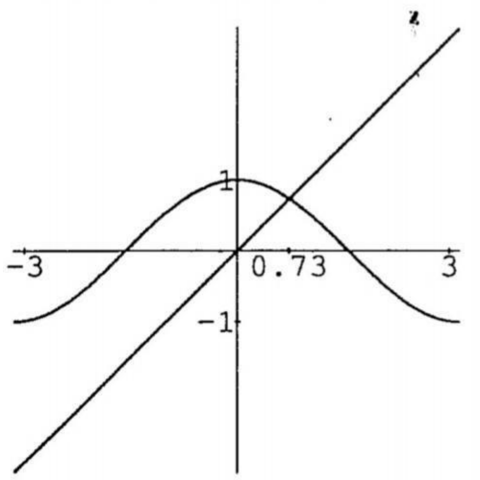

Figure 2: The fixed point of C(x) = cos x is 0.739085…

hand if $x_0 > 1$, then again the orbit moves far from 1. For example, the orbit of 1.1 is 1.1, 1.21, 1.4641, 2.1436, ... 21.114, ... Thus if $x_0 > 1$, we have $F^n (x_0) \rightarrow \infty$ as n tends to ∞ and hence the orbit tends far from 1. Clearly points that are close to 0 have orbits that are attracted to 0, while points close to 1 have orbits that are repelled from 1. To make the idea more clear , we can give the following definition.

Definition : Suppose x_0 is a fixed point for F. Then x_0 is an attracting fixed point if $|F^{'}(x_0)| < 1$. The point x_0 is a repelling fixed point if $|F^{'}(x_0)| > 1$.

For example consider, $F(x) = 2x(1 - x) = 2x - 2x^2$.

Clearly $x_0 = 0$ and $x_0 = 1/2$ are the fixed points for F. We have $F^{'}(x) = 2 - 4x$.

$F^{'}(0) = 2$ and $F^{'}(1/2) = 0$. Thus 0 is a repelling fixed point and 1/2 is an attracting fixed points.

PERIODIC ORBIT OR CYCLE

The point x_0 is periodic if $F^n(x_0) = x_0$ for some $n > 0$. The least such n is called the prime period of the orbit. If x_0 is periodic with prime period n ,then the orbit of x_0 is just a repeating sequence of numbers,

$$x_0, F(x_0), \ldots, F^{n-1}(x_0), x_0, F(x_0), \ldots F^{n-1}(x_0), \ldots$$

For example 0 lies on a cycle of prime period 2 for $F(x) = x^2 - 1$, since $F(0)= -1$ and $F(-1) = 0$. Thus the orbit of 0 is simply 0, -1, 0, -1, 0, -1,...... . We also say that 0 and -1 form a 2- cycle. Similarly 0 lies on a periodic orbit of prime period 3 or a 3-cycle for $F(x) = -3x^2/2 + 5x/2 + 1$, since $F(0)= 1$, $F(1) = -3/2 + 5/2 + 1 = 2$ and $F(2) = -3 \text{ x } 4/2+5\text{x}2/2+1=0$.So the orbit is 0, 1, 2, 0,1,2,

EVENTUALLY PERIODIC ORBIT

A point x_0 is called eventually fixed or eventually periodic if x_0 itself is not fixed or periodic, but some point on the orbit of x_0 is fixed or periodic. For example, -1 is eventually fixed for $F(x) = x^2$ since $F(-1) = 1$, which is fixed. Similarly 1 is eventually periodic for $F(x) = x^2 - 1$ since $F(1) = 0$, which lies on a cycle of period 2. The point $\sqrt{2}$ is also eventually periodic for this function, since the orbit is $\sqrt{2}$, 1, 0, -1, 0, -1, 0,.....

CHAPTER 2

CHAOS THEORY

Chaos theory is a scientific principle describing the unpredictability of systems. Most fully explored and recognised during the mid to late 1980's, its premise is that systems sometimes reside in chaos without any predictability or direction. These complex systems may be weather patterns, ecosystems, water flows, anatomical function or organisations. While these system's chaotic behaviour may appear random at first, chaotic systems can be defined by mathematical formula, and they are not without order or finite boundaries.

During the early 1960's a few scientists from various disciplines were taking note of odd behaviour in complex systems such as the earth's atmosphere and the human brain. One of these scientists was Edward Lorenz, a meteorologist from the Massachusetts Institute of Technology (MIT), who was experimenting with computational models of the atmosphere. In the process of his experimentation he discovered one of chaos theory's fundamental principles - The Butterfly Effect. The Butterfly Effect is named for its assertion that a butterfly flapping its wings in Tokyo can impact the weather patterns of Chicago. More scientifically, the Butterfly Effect proves that forces governing weather formation are unstable. These unstable forces allow minuscule changes in the atmosphere to have major impact elsewhere. More broadly applied, the Butterfly Effect means that what may appear to be insignificant changes to small parts of a system can have exponentially large effects on that system. It also helps to dispel the notion that random system activity and disturbances must be due to external influences, and not the result of minor fluctuations within the system itself. By the early 1980's, evidence accumulated that chaos theory was a real phenomenon. One of the first frequently - cited examples is a dripping water faucet. At times, water drops from a leaky faucet exhibit chaotic behaviour (water doesn't drip at a constant or orderly rate), eliminating the possibility of accurately predicting the timing of those drops. More recently the orbit of Pluto was shown to be chaotic. Scientists took advantage of applications using chaos to their benefits; chaos - aware control techniques could be used to stabilize lasers and heart rhythms, among multiple other uses.

DEFINITION OF CHAOS BY ROBERT L DEVANEY

There are many possible definitions of chaos. Before moving on to the definition given by Devaney, we shall discuss some important terminologies prerequisite to the definition.

1. Suppose X is a set and Y is a subset of X. We say that Y is dense in X if for any point $x \in X$, there is a point y in the subset Y arbitrarily close to x. Equivalently, Y is dense in X if for any $x \in X$ we can find a sequence of points $\{y_n\} \in Y$ that converges to x. For example, the a subset of rational numbers is dense in the set of real numbers. So is the subset consisting of all irrational numbers. Also (a,b) is dense in [a,b].

2. A dynamical system is transitive if for any pair of points x and y and any $\varepsilon > 0$ there is a third point z within ε of x whose orbit comes within z within ε of y. In other words a transitive dynamical system has the property that given any two points, we can find an orbit that comes arbitrarily close to both. Clearly, a dynamical system that has a dense orbit is transitive, for the dense orbit comes arbitrarily close to all points. The fact is that the converse is also true, i.e, a transitive dynamical system has a dense orbit. However we will not prove this fact since it uses an advanced result from real analysis.

3. A dynamical system F depends sensitively on initial conditions if there is a $\beta > 0$ such that for any x and any $\varepsilon > 0$ there is a y within ε of x and a k such that the distance between $F^k(x)$ and $F^k(y)$ is at least β.This says that, no matter which x we begin with no matter how small a region we choose about x, we can always find an y in this region whose orbit eventually separates from that of x by atleast β. Moreover the distance β is independent of x . As a consequence, for each x, there are points arbitrarily nearby whose orbits are eventually far from that of x.

Definition:

A dynamical system F is chaotic if

(a) Periodic points for F are dense

(b) F is transitive

(c) F depends sensitively on initial conditions.

DEFINITION OF CHAOS BY G. C. LAYEK

The mathematical definition of chaos introduces two notions, viz., the topological transitive property implying the mixing and the metrical property measuring the distance. Chaotic orbit may be expressed by fractals. Before defining chaos under the mathematical framework we discuss some preliminary concepts and definitions of topological and metric spaces which are essential for chaos theory.

(1) Let X be a nonempty set and $\eta \subseteq P(X)$, the power set of X. Then η is said to form a topology on X if

(i) the null set ϕ and the whole set X both belong to η ,

(ii) union of any collection of subsets of η belongs to η , and

(iii) intersection of finite collection of subsets of η belongs to η . If η is a topology on X, then the couple (X, η) is called a topological space. The subsets of η are called open sets. Some examples of topological spaces are given below:

(a) Let X be a nonempty set and $\eta = P(X)$. Then (X, η) forms a topological space. In this space, the topology η is called a discrete topology on X.

(b) Let X be a nonempty set and $\eta = \{\phi , X\}$. Then (X, η) forms a topological space. In this space, the topology η is called a trivial topology or an indiscrete topology on X.

(2). A metric space (X, d) contains a nonempty set X and a distance function $d : X \times X \rightarrow R$ such that for all x, y, z $\in$ X the following properties hold

(a) $d(x, y) = d(y, x)$ (symmetry)

(b) $d(x, y) = 0 \Longleftrightarrow x = y$ (identity)

(c) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

eg: Let X be a non empty set and d: $X \times X \rightarrow R$ be defined as $d(x, y) =0$ if $x = y$ and $d(x, y) = 1$ if x not equal to y $\forall$ x, y, z $\in$ X. Then (X, d) is a metric space. This metric space is known as discrete metric space.

(3). A dynamical system can be viewed as a couple (X, f) where f: $X\rightarrow X$ is a function from the topological space (or metric space) X into itself.

(4). Let f: $X\rightarrow X$ be a map. A set $A \subseteq X$ is said to be invariant under the map f if for any $x \in A$, $f^n(x) \in A$ $\forall$ n. Specifically the set A is invariant if $f(A) = A$. Let (X, f) be a discrete dynamical system. A subset A of X is said to be a positively invariant set if $f(A) \subset A$. If $f(A) = A$, then A is strictly positively invariant. The set of periodic points of a map is always an invariant set.

(5). In a topological space (X, η), a subset A of X is said to be a dense set (or an everywhere dense set) if $\bar{A} = X$. In otherwords, A is said to be dense subset of X iff for any $x \in A$, any neighborhood of x contains at least one point of A. For example, the set of all rational numbers is dense subset of the set of all real numbers.

(6). A map $f : X \to X$ is said to have sensitive dependence on initial conditions (SDIC) property if there exists a $\delta > 0$ such that for any $x \in X$ and any neighborhood $N_\epsilon(x) = (x - \varepsilon, x + \varepsilon)$ of x, there exist $y \in N_\epsilon(x)$ and an integer $k > 0$ such that the property $|f^k(x) - f^k(y)| > \delta$ holds good. Let us explain the concept by considering an example. The doubling map $g:S \to S$ on a unit circle S is defined by $g(\theta) = 2\theta$. Let $\theta_1 \in S$ and $N_\epsilon(\theta_1) = (\theta_1 - \varepsilon, \theta_1 + \varepsilon)$ be an nbd of θ_1. Let $\delta > 0$ then there exists $\theta_2 \in N_\epsilon(\theta_1)$ and $k > 0$ such that $|g^k(\theta_1) - g^k(\theta_2)| = |2^k\theta_1 - 2^k\theta_2| = 2^k|\theta_1 - \theta_2| > \delta$, for all $\theta_1, \theta_2 \in N(\theta_1)$. This implies that the map g has sensitive dependence property.

(7). Transitivity is one of the fundamental property in the mathematical theory of chaos. A map $f : X \to X$ is said to be topologically transitive on X if for any two open sets $U, V \subset X$ there exists $k \in N$ such that $f^k(U) \cap V \neq \phi$.

The function f is totally transitive when the composition function is topologically transitive for all integers $n \geq 1$. A topologically transitive map has fixed points which eventually move under iterations from one arbitrarily small nbd to the other. Hence, the orbit cannot be decomposed into two disjoint open sets which are invariant under the map. A discrete dynamical system is decomposable if there exists a finite open cover (with at least two elements) of X such that each open set of the cover is positively invariant under the map f. On the other hand, the system is indecomposable if and only if it cannot be expressed as the union of two nonempty, closed, and positively invariant subsets of X. Thus the topological transitivity implies indecomposobility.

(8). Topological mixing is a stronger notion of topological transitivity. A map $f : X \to X$ is said to be strongly transitive if for every $x, y \in X$ we can find at least one point z very near to x that moves under iterations to small nbd of y. A map $f : X \to X$ is said to be topologically mixing on X if for any two open sets, $U, V \subset X$ with $U \cap V \neq \phi$, there exists a positive integer N such that $f^n(U) \cap V \neq \phi$ for all $n \geq N$.

Definition: A map $f : X \to X$ (X is either a topological space or a metric space) is said to be a chaotic map on an invariant subset $A \subseteq X$ if the following conditions are satisfied:

(i) the map f has sensitive dependence on initial conditions on A.

(ii) f is topologically transitive on A.

(iii) the periodic points of f are dense in A.

Even though we have discussed two definitions, we follow the definition given by Robert L Devaney.

SHARKOVSKII'S THEOREM

In 1964, the the Russian (Ukrainian) mathematician, A.N. Sharkovskii in his paper "Coexistence of Cycles of a Continuous Map of a line into itself," published in Ukrainian Mathematical Journal (1964), proved a remarkable theorem in discrete dynamical system. According to his name the theorem is called Sharkovskiis theorem. The theorem plays an important role in verifying the existence of periodic cycles of certain periods of a one-dimensional real-valued map from the existence of periodic cycles of different periods of the map. Sharkovskki's theorem is an incredibly powerful and beautiful strengthening of the Period 3 Theorem. Period 3 Theorem states that "Suppose $F : R \rightarrow R$ is continuous and also F has a periodic point of prime period 3. Then F also has periodic points of all other periods."

The common well-known order in natural numbers is 1, 2, 3, 4, 5, 6,7, 8, 9, ... To state the Sharkovskii's theorem we first list all of the natural numbers in the following strange order :

3, 5, 7, 9, ...

2.3, 2.5, 2.7, 2.9, ...

$2^2.3$, $2^2.5$, $2^2.7$, $2^2.9$, …

$2^3.3$, $2^3.5$, $2^3.7$, $2^3.9$, ...

....

....

...2^n, ..., 2^3, 2^2, 2, 1

This is known as the Sharkovskii's ordering of natural numbers and it is constructed as follows. First list all positive odd integers greater than 1 in increasing order. Then list the integers that are 2 times the odd integers greater than 1 in the increasing order and then the numbers which are 22 times of the odd integers and so on.

Finally list the integers in decreasing order that are the integral powers of 2. In the Sharkovskii's ordering, the integer 3 is the smallest number and 1 is the largest number.

Sharkovskii's Theorem: Suppose $F : R \to R$ is continuous. Suppose that F has a periodic point of period n and that n precedes k in the Sharkovskii's ordering. Then F also has a periodic point of prime period k.The theorem is also true if $F : I \to I$ is continuous, where I is a closed interval of the form [a, b]. We simply extend F to the entire real line by defining $F(x) = F(a)$ if $x < a$ and $F(x) = F(b)$ if $x > b$. The resulting extension is clearly a continuous function. Also the Period 3 Theorem is an immediate corollary of Sharkovskii's theorem.

SHIFT MAP

The Shift map $\zeta : \Sigma \to \Sigma$ is defined by $\zeta(s_0s_1s_2...) = (s_1s_2s_3...)$. That is ζ simply drops the first entry of any point in Σ.

For example: $\zeta(010101...) = (101010...)$

$\zeta(01111...) = (1111...)$

Periodic points of the Shift map: A point p is said to be a periodic point of period n of a map f if $f^n(p) = p$. The least positive integer n for which $f^n(p) = p$ is called the prime period of the periodic point p. The periodic points of a map generate a repeating sequence. This sequence can also be observed for the shift map. For example, consider a point $p_n = (s_0s_1s_2...s_{n-1}s_0s_1...s_{n-1}...)$ in Σ. Applying the shift operator ζ to p_n repeatedly we see that,

$\zeta(p_n) = (s_1s_2\ldots s_{n-1}s_0s_1\ldots s_{n-1}\ldots)$

$\zeta^2(p_n) = (s_2s_3\ldots s_{n-1}s_0s_1\ldots s_{n-1}\ldots)$

....

....

....

$\zeta^{n-1}(p_n) = (s_{n-1}s_0s_1\ldots s_{n-1}\ldots\ldots\ldots)$

$\zeta^n(p_n) = (s_0s_1s_2\ldots s_{n-1}s_0s_1\ldots s_{n-1}\ldots) = p_n$

This shows that p_n is a periodic point of the shift map ζ of prime period n.

Theorem: The Shift map $\zeta : \Sigma_m \rightarrow \Sigma_m$ is chaotic.

Proof: To prove that ζ is chaotic, it is enough to prove ζ is topologically transitive, it has sensitive dependence on initial conditions and the set of all periodic points of ζ is dense in Σ_m.

We have " A topological dynamical system $f : X \rightarrow X$ is topologically transitive if for every pair of non empty open sets U, V of X, there exists $n \in N$ such that $f^n(U) \cap V \neq \phi$." So in order to show that ζ is topologically transitive we need to prove that for any two non empty open sets U and V of Σ_m, there exists $n \in N$ such that $\zeta^n(U) \cap V \neq \phi$.

For, let U and V be any two open sets of Σ_m. Let $x = (x_1, x_2, x_3, \ldots) \in U$ and $y = (y_1, y_2, y_3, \quad) \in V$. Then there exists open balls $B(x, r_1) \subseteq U$ and $B(y, r_2) \subseteq V$. If $r = \min\{r_1, r_2\}$, then $B(x, r) \subseteq U$ and $B(y, r) \subseteq V$. We choose $n \in N$

such that $1/m^n < r$. Consider the point $z = (x_1, x_2, x_3, ...x_n, y_1, y_2, ...) \in \Sigma_m$ which agrees with x upto the n^{th}.

Therefore, by proximity theorem, we have $d(x, z) \leq 1/m^n < r$.

$\Rightarrow z \in B(x, r) \subseteq U$ and consequently it follows that $\zeta^n(z) \in \zeta^n(U)$. Also $\zeta^n(z) = (y_1, y_2, ...) = y \in V$, so

$y = \zeta^n(z) \in \zeta^n(U) \Rightarrow y = \zeta^n(z) \in \zeta^n(U) \cap V$.

Therefore, $\zeta^n(U) \cap V \neq \phi$.

Therefore, ζ is topologically transitive.

Now we need to show that ζ has sensitive dependence in initial conditions. For let $x \in \Sigma_m$ be arbitrary and N(x) be an arbitrary nbd of x. Then by definition of nbd, there exists a non empty open set G such that $x \in G \subseteq N(x)$. Now, $x \in G$, G is open in Σ_m implies there exists an open ball B(x, r) such that $B(x, r) \subseteq G \subseteq N(x)$. Let $y \in B(x, r) \subseteq G \subseteq N(x)$ such that $x \neq y$ and x is very close to y. It is always possible to have a very close point to x, because we can choose a $k \in N$ as large as we want satisfying $1/m^k < r$ and for this large $k \in N$ we can construct the point y in such a way that this agrees with x up to k digits. Then $d(x, y) \leq 1/m^k < r$ and hence for large value of k, x will be too close to y. Let $d(x, y) = \varepsilon$. Then, since x is very close to y and ε is very small, so depending on the value of $\varepsilon > 0$, there exists a large and unique $n \in N$ such that $1/m^{n+1} < \varepsilon \leq 1/m^n$. Consider

$d(x, y) = \varepsilon \leq 1/m^n$

Then $d(x, y) \leq 1/m^n \Rightarrow$ x and y agrees up to the n^{th} digit

$\Rightarrow$ (n+1)th digits of x and y are different.

$\Rightarrow$ The first digit of $\zeta^n(x)$ and $\zeta^n(y)$ are different.

$\Rightarrow d(\zeta^n(x), \zeta^n(y)) \geq 1/m$. Here, from the above relation it is clear that 1/m plays the role of sensitivity constant δ.Thus for every $x \in \Sigma_m$ and any nbd N(x) of x, there exists $y \in N(x)$ and n>0 satisfying $d(\zeta^n(x), \zeta^n(y)) \geq \delta$ for $\delta = 1/m$. Thus ζ has sensitive dependence on initial conditions.

Now it only remains to prove that the set of all periodic points of ζ is dense in Σ_m. For that we first show that ζ has m^n- m periodic points of period - n in Σ_m for $n \geq 2$. It is to be noted that if a definite block of n - digits from the set {0, 1, 2, ... m-1} repeats indefinitely, then it is a periodic point of ζ of period - n in Σ_m. A block of n - digits can be formed with the m distinct digits 0, 1, 2, ..., m-1 in m^{n-1} ways. These blocks contain the m - blocks formed by the same digit (eg: 000000000.....00,1111111.....11, 22222222 22, etc) which are not periodic points of period-n. These are in fact, periodic points of period-1 i.e, fixed points. So, we have only (m^n- m) numbers of periodic points of period - n in Σ_m. Consider an arbitrary point $x \in \Sigma_m$. We show that for any $\varepsilon > 0$, however small,there is a point $p \in P(\zeta)$ such that $d(x, p) < \varepsilon$. Let $x = (x_1, x_2, x_3, ...)$. For the fixed small $\varepsilon > 0$, we can always find a positive integer $n \in N$ such that $1/m^n < \varepsilon$. Now we construct $p \in P(\zeta)$ of period (n+1) such that $p =(x_1, x_2, x_3, \quad x_n, y, x_1, x_2, x_3, ...x_n y, x_1, x_2, x_2, ...x_n, y, ...)$ i.e, p is constructed by repeating $(x_1, x_2, x_3, \quad x_n, y)$ infinite number of times so that it agrees with the digits of x up to n- terms and disagrees at $(n + 1)^{th}$ digit such that $x_{n+1} \neq y$ and $d(x, p) \leq 1/m^n < \varepsilon$. Thus, for every $x \in \Sigma_m$ and $\varepsilon > 0$, there exists $p \in P(\zeta)$ such that $d(x, p) < \varepsilon$. That is, however small $\varepsilon > 0$ may be,for any $x \in \Sigma_m$ there is always a point $p \in P(\zeta)$ which is at a distance less than the arbitrarily small quantity $\varepsilon > 0$. Hence the set $P(\zeta)$ is dense.

TENT MAP

The Tent Map is a one dimensional piecewise linear map. Its graph resembles the front view of a tent. It is also called a triangle map or a stretch and fold map. Generally, the Tent map $T: [0,1] \rightarrow [0,1]$ is defined as $T(x) = 2\lambda x$ if $0 \leq x \leq 1/2$ and $T(x) = 2\lambda(1 - x)$ if $1/2 \leq x \leq 1$, where λ $(0 \leq \lambda \leq 1)$ is a control parameter.

Theorem: The Tent map is chaotic on [0,1].

Proof: To prove T is chaotic on [0,1], it is enough to prove that T has sensitive dependence on initial conditions, T has dense set of periodic orbits and T has at least one dense orbit i.e, T is topologically transitive.

Let $x \in [0,1]$. First we will show that if v is any dyadic rational number

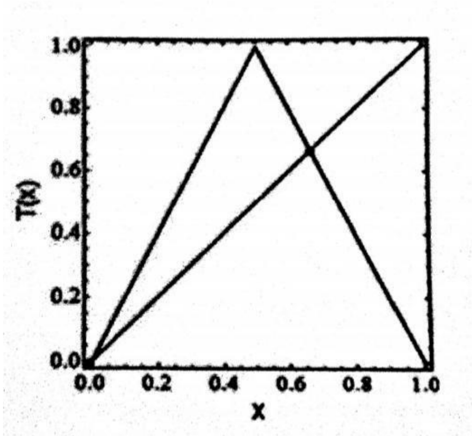

Figure 3:

(of the form $j/2^m$, in the lowest terms) in [0,1] and w is any irrational number in [0,1], then there is a positive integer n such that $|T^{(n)}(v)-T^{(n)}(w)|> 1/2$. Toward that goal, if $v = j/2^m$, then $T^{[m]}(v) = 1$ and $T^{[m+k]}(v) = 0$ for all $k> 0$. By contrast, if w is any irrational number in [0,1], then T doubles each number in (0, 1/2), there exists $n > m$ such that $T^{[n]}(w) > 1/2$. Since $n > m$, it follows that $T^n(v) = 0$. Next let $\delta > 0$, then there exists a dyadic rational v and an irrational number w in [0,1] such that $|x - v| < \delta$ and $|x-w| < \delta$. Therefore either $|T^{(n)}(x)-T^{(n)}(v)| > 1/4$ or $|T^{(n)}(x)-T^{(n)}(W)| > 1/4$. Thus if we let ε, then sensitive dependence on initial conditions at the arbitrary number x, and hence on [0,1] is proved.

Let (a,b) be any open interval in (0,1). Let p be an odd positive integer large enough so that 1/2.p < b-a. There is a least positive integer k such that k/p ∈ (a, a+b/2). If k is even, then we are done. If k is odd then k+1/p ∈ (a,b) will do. Thus the set of periodic points under T is dense in [0,1]. Thus T is chaotic.

Chapter 3

FRACTALS AND THEIR PROPERTIES

With the advent of civilization, the human mind always tends to unravel the wealth of knowledge in nature, whether it is his curiosity to know the universe or to measure the length of the coastlines of the earth. However,despite discovering modern technological tools, most of the knowledge remains unknown. Nature possesses objects that are irregular and erratic in shape. With the aid of Euclidean geometry it became possible to give detail descriptions of length, area, volume of objects like line, square, cube, etc.,but for many natural objects like cauliflower, leaves patterns of trees, shape of mountains, clouds, etc., the idea of length, area remains vague until Benoit Mandelbrot (1924 - 2010). In the year 1975, he introduced a new branch of geometry known as fractal geometry which finds order in chaotic shapes and processes, and also wrote a book on The fractal geometry of nature in 1977.More specifically, fractal geometry describes the fractals, which are complex geometric structures prevalent in natural and physical sciences.

In the earliest civilizations these complex structures now known as fractals were regarded as formless, consequently the study of fractals was rejected until Mandelbrot who claimed that these formless objects can be described by fractal geometry. Most fractal objects are self-similar in nature. By self-similarity we mean that if a tiny portion of a geometric structure is magnified, an analogous structure of the whole is obtained. The self-similarity does not always mean the usual geometric self-similarity in which self-similarity of the shape is considered, but it may also be statistical in which the degree of irregularity or fragmentation of the shape is same at all scales. However the converse is not true, that is not every self-similar object is a fractal. The geometry of fractal is different from Euclidean geometry. Among the differences first comes the dimension of the objects, the objects of Euclidean geometry always have an integer dimension like line has dimension 1, square has dimension 2 and a cube has dimension 3. Wherein fractal objects usually have fractional dimension for instance 1.256, even though there are exceptions such as the Brownian motion, Peano curve, which have fractal dimension 2 and devils staircase has fractal dimension 1.

The perimeter and surface area of a fractal object is not unique like the regular objects and changes its values with finer resolution. Thus the perimeter and area of fractals are undefined; this means that the object cannot be well approximated with regular geometry like square and cubes of Euclidean geometry. Natural objects like ferns, trees, snowflakes, seashells, lightning bolts, cauliflower, or broccoli are fractals. The natural processes which grow with the revolution of time such as sea coast, surface of moon, clouds, mountains, veins, and lungs of humans and animals are all approximately described by fractal geometry. As we know chaotic orbits are highly irregular. Fractals are useful to study chaotic orbits and may be represented by fractals. Fractals are not just a matter of geometry but have a number of applications for the well-being of life. Fractal properties are useful in medical science.

FRACTALS

The name fractal was invented by Polish-French-American mathematician Benoit Mandelbrot in the year 1975, from the Latin adjective fractus, which means broken or fractured. Mandelbrot created a beautiful fractal represented by a set of complex numbers named after him as Mandelbrot set. Since then he has been regarded as the originator of fractal geometry and fractals. However, at that time there already existed some mathematically developed fractals such as Cantor set by George Cantor in 1872, Peanos curve by Giuseppe Peano in 1890, Koch's curve by Helge von Koch in 1904, Sierpinskys fractals such as carpet, triangle, etc., by Waclaw Sierpinski in 1916, Julia set by Gaston Julia in 1918 but Mandelbrot was the man who gave all of these structures a common name and a tool to describe the properties and complexities underlying these irregular and erratic structures.

However, there is a huge difference between the natural fractals and mathematically developed fractals because natural fractals are always growing with time, they are generally dynamical processes while those developed mathematically are regarded as static (do not change with time). Fractal is an object that appears self-similar with varying degrees of magnification. It does not have any characteristic length scale to measure and details of its structure would reveal if looked at finer resolution. Furthermore, it possesses symmetry across the scale with each small part of the object reproducing the structure of the whole.The fundamental difference between fractal and non-fractal objects is that when a non-fractal object is magnified, it cannot reveal the original feature. For instance, if a section of an ellipse is magnified, it loses its feature of being an ellipse (shown in Fig. 13.1); but a fractal object always reveals its original feature under magnifications.

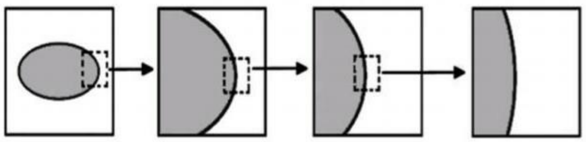

Figure 4: A non fractal object (ellipse).

The main properties of fractal objects are:

(i) Fractals appear with some degrees of self-similarity that is, if a tiny portion of a fractal object is enlarged, features reminiscent of the whole will be discerned. This means that the fractal objects could be broken into even finer pieces having same features as the original;

(ii) The dimension of a fractal object is usually not an integer but fractional;

(iii) Usually, the perimeter and area of a fractal object is undefined with changing values depending upon the resolution taken to measure it. In some cases, the perimeter is immeasurable (being infinite), but area is finite;

(iv) Fractal geometry can be represented by an iterative algorithm for instance the Mandelbrot set is the set of points (say z_0) in the complex plane for which the iteration is given by $z_{n+1} = z^2_n + z_0$ which remains bounded. Another example is the Julia set represented by the function $f(z) = z^2 + c$.

SELF SIMILARITY AND SCALING

Self-similarity and scaling are two bases for describing fractal geometry. Basically, the concept of self-similarity is an extension of the concept of similarity in mathematics or physical processes. Two objects are said to be similar if they are of the same shape, regardless of their size. Normally, the similarity between two objects or among objects is studied under the light of transformations. Two objects are said to be similar if one is obtained from the other by a transformation, known as similarity transformation. Similarity transformations are the combinations of translation, rotation, and scaling.

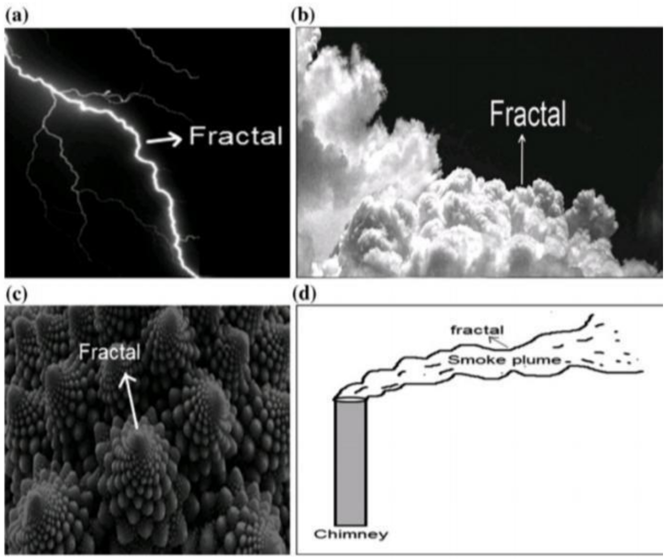

Figure 5: Some fractal objects: (a) forked lighting in the sky, (b) cloud boundaries, (c) broccoli, (d) smoke coming out of chimney.

We shall now discuss these transformations in detail in R^2

Let P(x, y) be any point in R^2 . A translation operator denoted by T, translates the point P into a new point P´(´x, ´y) by the rule (´x, ´y) = T(x,y) =(x+c, y+d), that is ´x = x+c and ´y = y+d where c and d are real numbers. Here c represents the displacement of the point P in the horizontal direction(x-axis) and d represents the displacement of the point P in the vertical direction (y-axis). For example consider the rectangle ABCD as shown in the figure.

Taking c = -1 and d = 1, we can find the translated rectangle A´B´C´D´ as shown below :

Thus under translation T the shape and size of an object remain invariant.

A scaling operation S takes the point P(x, y) to a new point P′(′x, ′y) by the formula (′x, ′y) = T(x, y) = (αx, αy) i.e, ′x = αx and ′y = αy, where α> 0 is a real number, known as the scaling factor. Under the scaling operation S, an object will either contract or expand. For α < 1, it contracts and

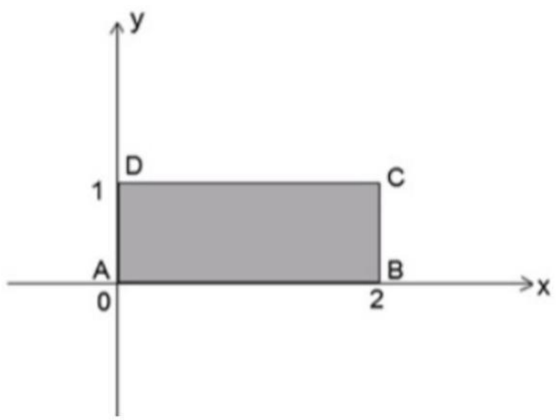

Figure 6: Rectangle ABCD in the XY plane

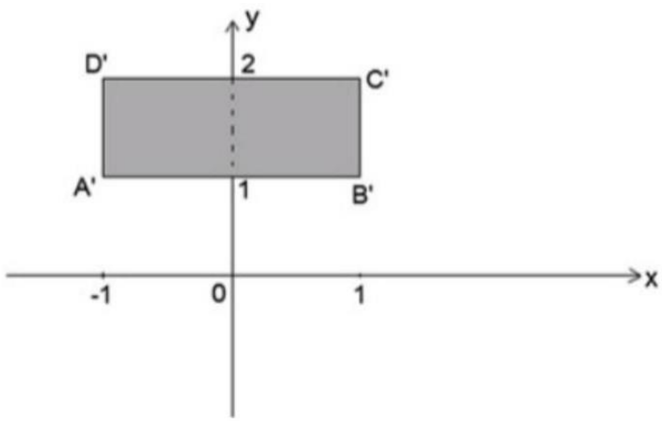

Figure 7: Translation of the rectangle ABCD-1 unit along the x -axis and 1 unit along the y -axis

for α > 1, it expands. Consider the rectangle in the previous example, for the scaling factor α = 2 and α = 1/2, the rectangle ABCD reduces to the rectangle A´B´C´D´ and ´A´B´C´D´ respectively.

Figure 8:

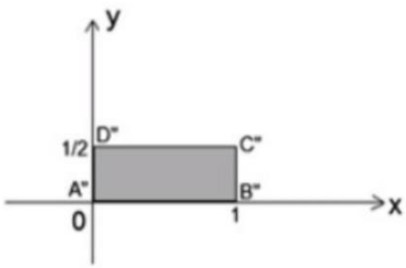

Figure 9:

This illustrates that the transformed figures are associated with some transformations. We will now try to understand the concept of self similarity by the following example of the self similarity of a square. Consider the following self similar square. This square can be divided into 4 equal squares as shown below. Taking any one of these four squares and magnifying (scale) it by the scaling factor $\alpha = 2$, we will obtain the original square. This is actually the property of self similar objects. Self-similarity usually does not mean that a magnified view is identical to the whole object, but instead the character of patterns is same on all scales. Normally, self-similarity of fractal objects, natural, physical, and biological processes may occur in different types, viz., self-similarity in space, self-similarity in time, and also statistical self-similarity. We shall illustrate these different types of self-similarities as follows:

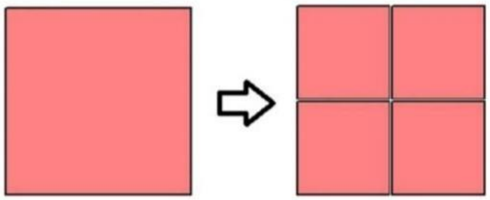

Figure 10: Self similarity of a square

1. Self similarity in space: If the pattern of an object or dynamical process in large structures obtains under the repetition of smaller structure then the similarity exhibited by the object is called as self-similarity in space. The measurement of area of an object depends on the length of the space used to measure it. The length of the space gives the spatial resolution of the measurement. For finer resolution, area of the object gets increased. For examples, the self-similar patterns in biological systems such as the arteries and the veins in the retina and the tubes that bring air into the lungs maintain the self-similarity in space. The self similarity in space had been better described by Mandelbrot in his 1967 paper entitled How long is the coastline of Britain, based on measuring the length of coastlines first considered by British meteorologist Lewis Fry Richardson, who tried to measure the length of the coastline of Britain by laying small straight line segments of the same length, end to end, along the coastline. The spatial resolution of the measurement is set by the length of these line segments and the combined length of these line segments give the total length of the coastline. He observed that the length of coastline

increased each time when the length of the line segment measuring the coastline decreased, i.e., whenever the measurement was taken at ever finer resolutions since the smaller line segments included the smaller bays and peninsulas that were not included in the measurement of larger line segments. The total length of coastlines thus varies with the length of line segment measuring it.Coastlines are approximately described by fractal geometry. Thus the length of fractals is immeasurable.

2. Self similarity in time: If the pattern of the smaller fluctuations of a process over short times is repeated in the larger fluctuations over longer times then the self-similarity in the process is found in terms of time and is known as self-similarity in time. The electrical signals generated by the contraction of the heart, the volumes of breaths over time drawn into the lung, etc. are some examples of self-similarity in time.

3. Statistical self similarity: When the statistical properties such as mean, variance, correlations, etc., of the smaller pieces are exactly similar to the statistical properties of the larger pieces, then the self-similarity is called as statistical self-similarity. We can say that just like geometrical structure satisfies a scaling relation so does the stochastic process. For instance, consider the cloud boundaries. They are rugged (uneven or irregular) in nature. If a small portion of the cloud boundary is enlarged it looks approximately the same (same degree of rugedness).

It is not exactly self-similar. Actually, the smaller copies of the cloud boundaries are approximately the same kind of their larger portions, but some of the statistical properties remain invariant. This type of self-similarity is known as statistical self-similarity. The blood vessels in the retina, the tubes that bring air into the lungs, the electrical voltage across the cell membrane, wall cracks, sea coast lines, and ECG report of heartbeat of a normal man possess the statistical self-similar property.

SELF SIMILAR FRACTALS

Fractal objects are omnipresent in nature and can be well-approximated mathematically. Mandelbrot set, Koch snowflake, and Sierpinski triangle are all examples of mathematically generated fractals. All fractal objects in nature are self-similar at least approximately or statistically if not exactly. Natural objects are usually approximately fractal, and we are well aware of the fact that one cannot find any natural object or phenomena which are exactly self-similar. In contrast the mathematical fractals are perfect fractals where each smaller copy is an exact copy of the original.

The Sierpinski triangle S may be considered as the composition of three small equal triangles, each of which is exactly half the size of the original triangle S. Thus if we magnify any of these three triangles by a factor of 2, we will get the triangle S. So, the triangle S consists of three small triangles, which are self-similar copies of S. Again, each of these three small self-similar triangles can be considered as the combination of another three small self-similar triangles, each of which is the scaled-down copy with scaling factor $\alpha = 1/2$ and so on. Thus we see that if we enlarge a small portion of the Sierpinski triangle by a suitable scaling factor. it will give the original feature. This type of self-similarity is known as exact self-similarity. It is the strongest self-similarity occurred in fractal images. Likewise Sierpinski gasket, Cantor set, von Koch curve, fern tip, Menger sponge, etc., possess exact self-similarity property. It is to be noted that most of the fractal objects are self-similar, but a self-similar object is not necessarily a fractal. For example, a solid square can be divided into four small solid squares that resemble the original large square, and each small square can be divided into four smaller solid squares resembling the large square, and so on. Hence a square is self-similar. But it is not a fractal, and is a regular geometric shape whose properties can be well described by the Euclidean geometry.

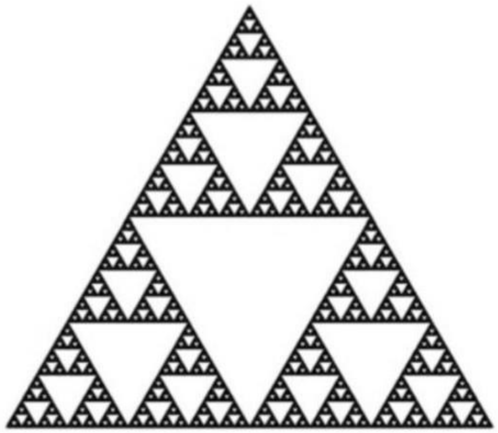

Figure 11: Sierpinski triangle

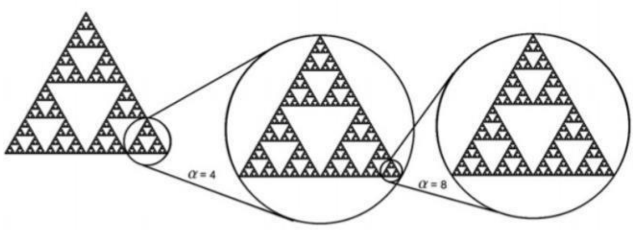

Figure 12: Magnification of a portion of the Sirepinski triangle first by scaling factor 4 and then by factor 8.

CONSTRUCTION OF VON KOCH CURVE

The Swedish mathematician Helge von Koch (1870 - 1924) introduced the Koch curve in the year 1904, as an example of a curve that is continuous but nowhere differentiable. We know that tangent at a corner of a curve is not uniquely defined. The Koch curve is made out of corners everywhere, therefore it is not possible to draw tangent at any of its points hence nowhere differentiable. This curve can be constructed geometrically by successive iterations as follows.

We start with a line segment, say, S_0 of length L_0. To generate S_1, divide it into three equal line segments. Then replace the middle segment by an equilateral triangle without a base. This completes the first step (S_1) of the construction, giving a curve of four line segments, each of length $l = L_0/3$ and the total length is $4L_0/3$ in this stage of construction. To generate S_2, remove the middle line segment of each of the above 4 line segments and replace by equilateral triangle without base. This completes the second step (S_2) of the construction, giving

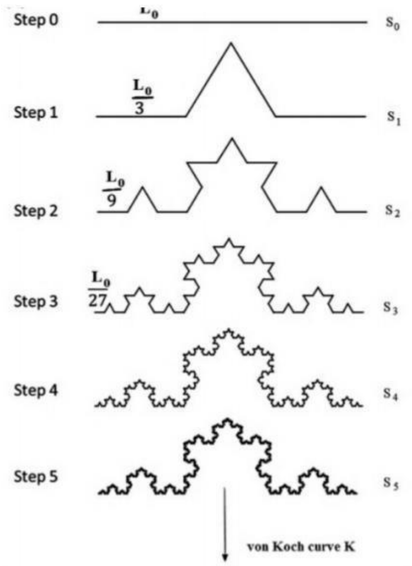

Figure 13: Construction of von Koch curve

a curve of 16 line segments, each of length $l = L_0/ 32$ and the total length is $16L_0/32$ in this stage of construction. We repeat the process again and again and hence at the nth stage number of copies is equal to $N = 4^n$ line segments, each of length $L_0/3^n$. The limiting set $K=S\infty$ is known as the von koch curve. The length increases by a factor of 4/3 at each stage of the construction. So at the nth stage the length of the segments are given by $S_n = 4^n L_0/3^n = L_0 (4/3)^n$. So $S_n \rightarrow \infty$ as $n \rightarrow \infty$. Hence the length of the von koch curve is infinite.

CONSTRUCTION OF SIERPINSKI TRIANGLE

It was discovered by the Polish mathematician Waclaw Sierpinski (1882 - 1969) in 1916. Consider a solid (filled) equilateral triangle, say S_0 ,with each side of unit length. Now divide this triangle into four equal small equilateral triangles using the mid- points of the three vertices of the original triangle S_0 as new vertices. Then remove the interior of the middle triangle. This generates the stage S_1 . Repeat this process in each of the remaining three equal solid equilateral triangles to produce the stage S_2 . Repeat this process continuously for further evolution and finally the Sierpinski triangle is formed.

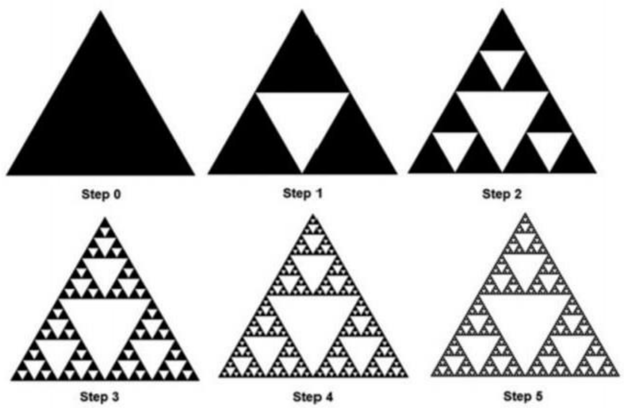

Figure 14: Construction of Sierpinski triangle

S_1 is covered by N = 3 small equal equilateral triangles each of side $\varepsilon = 1/2$. Similarly S_2 is covered by $N = 3^2$ triangles of side $= 1/2^2$. In general, S_n at the nth stage is covered by $N = 3^n$ triangles of side $\varepsilon = 1/2^n$. The area at the nth stage is obtained as $3^n\sqrt{3}/4(\ 1/2)^{2n} = \sqrt{3}/4(\ 3/4)^n \to 0$ as $n \to \infty$.

JULIA SETS

It was discovered by Gaston Julia (1893-1975) in the year 1918. Julia set represents the transformation function which is either a complex polynomial or complex rational function from one state of the system to the next, and is a source of the majority of attractive fractals known at present. Julia set is obtained by iterating the quadratic function $f(z) = z^2 + c$ for a complex initial value say z_0 , where c is an arbitrary fixed complex constant. Thus, by fixing the value of c and taking an initial value z_0 of z, one will obtain $f(z) = z^2_0 + c$ after the first iteration. The next iteration will give $f(z) = (z^2_0 + c)^2 + c$. Thus for a fixed value of c, the successive iterations give a sequence of complex numbers, i.e., $z^2_0 + c$, $(z^2_0 + c)^2 + c$, $((z^2_0 + c)^2 + c)^2 + c$ and so on. This sequence is either bounded or unbounded. Actually, Julia set is the boundary set between two mathematically different sets, escape set say E and the prisoner set say P. The escape set is the set of all those initial points z_0 for which the iterations give an unbounded sequence of complex number which escapes any bounded region and the prisoner set P is the collection of remaining initial points for which the iteration remains in a bounded region for always. Thus the complex plane of initial values is subdivided into two subsets E and P, and the Julia set is the boundary between them. However, one should be careful in regarding any boundary set like this as fractals. For instance, if D denotes a disk with center at 0 and radius 1 and that E is the region outside the disk. Then the boundary between E and D is the unit circle. The definition of Julia set suggests that the unit circle is a Julia set, but being a regular geometric shape it is not a fractal. Now carrying out iterations of f(z) fixing different values of c would yield different structures of Julia set. Therefore, there is a possibility of existence of an infinite number of such beautiful fractals. Mathematically, the escape set and the prisoner set can be defined as follows The escape set for the parameter c is given by

$E_c = \{ z_0 : |z_n| \rightarrow \infty \text{ as } n \rightarrow \infty \}$. The prisoner set for parameter c is given by $P_c = \{ z_0 : z_0 \text{ does not belong to } E_c \}$. Both escape set E and the prisoner set P fill some part of the complex plane and complement each other. Thus, the boundary of the prisoner set is simultaneously the boundary of the escape set, which is the Julia set for c.

3.6 MANDELBROT SET

Mandelbrot set is attributed to Benoit Mandelbrot, who discovered this set in 1979. It is the region in the complex plane comprising the values of c for which the trajectories defined by $Z_{k+1} = Z^2_k + c$; k = 0,1,2,. remains bounded for $k \rightarrow \infty$. We know that the Julia set and the prisoner set are either connected or totally disconnected. Mandelbrot set consists of those values of c for which the Julia set (J_c) is connected, i.e., $M = \{c \in C : J_c \text{ is connected}\}$. Mandelbrot set represents an extremely intricate structure. It is made up of a big cardiode to which a series of circular buds are attached. Each of these buds is surrounded by further buds and so on. From each bud there grows a fine branched hair in the outer direction. If these hairs are viewed at enlarged magnification one will find Mandelbrot sets that are self-similar with the actual Mandelbrot set. Mandelbrot set contains

Figure 15: Mandelbrot set

an enormous amount of information about the structure of Julia set as when the boundaries of Mandelbrot set are magnified an infinite structures of Julia sets are revealed for some values of c.

CHAPTER 4

APPLICATION

Fractals are not just complex shapes and pretty pictures generated by computers. Any thing that appears random and irregular can be a fractal. Fractals permeate our lives, appearing in places as tiny as the membrane of a cell and as majestic as the solar system. Fractals are the unique, irregular patterns left behind by the unpredictable movements of the chaotic world at work. In theory one can argue that every thing existing in this world is a fractal; for example the branching of the tracheal tubes, the leaves of trees, veins in a hand, water swirling an twisting out of a tap, a puffy cumulous cloud, tiny oxygen molecule or the DNA molecule, all of these are fractals.

Fractals have always been associated with the term chaos. One author elegently describes fractals as the patterns of chaos. Fractals depict chaotic behaviour, yet if one looks closely enough, it is always possible to get glimpses of self-similarity within a fractal. To many chaologists, the study of chaos and fractals is more than just a new field in science that unifies mathematics, theoretical physics, art and computer science- it is a new revolution. It is the discovery of a new geometry, one that describes the boundless universe we live in; one that is in constant motion, not as static images in textbooks. Today many scientists are trying to find applications for fractal geometry from predicting stock market prices to make new discoveries in theoretical physics. Fractals have more and more applications in science. The main reason is that they very often describe the real world better than traditional mathematics and physics.

CHAOS IN CARDIOLOGY

The human heart also has a chaotic pattern. The time between beats does not remain constant. It depends on how much activity a person is doing, among other things. Under certain conditions, the heart beat can speed up. Under different conditions, the heart beats erratically. It might even be called a chaotic heart beat. The analysis of a heart beat can help medical researchers find way to put an abnormal heartbeat back into a steady state, instead of uncontrolled chaos. However more sensitive instruments reveal that normal heart rhythm shows small vari-ability in the interval between beats. Our hearts rarely beat the same way twice. Varying opinions on the role of randomness and chaos have been proposed, among which one standing ground is that heart function is non chaotic when healthy and turns erratic, with creation of spatial chaos. Also reports say that even in normal state, heart physiology and function is actually chaotic and when these attributes become less random or chaotic, cardia disfunction manifests and death occurs.

CHAOS IN SURGERY

In surgery, N - plastery, W - plastery and M - plastery techniques have been in vogue for long. Irregular lengthening of an incision or scar has been seen to produce better results, both surgically and aesthetically. Why

does a lacreation heal better when its edges jagged or serrate and not alligned straight and side by side has been a question never raised or answered by the pathologists or surgeons. Unknowingly plastic surgery has been applying the techniques described as chaos systems. The reason why N, W or M procedures produce less scar tissue could be due to the operation of chaos theory. The major irregular the input,
the healthier the output.

CHAOS IN PHYSIOLOGY

According to some, even cyclical and periodic physiological processes such as menstruation including the transition to menopause, results from a specififc kind of complex system namely, one that is non linear, dynamical and chaotic. The dynamics of fluid flow and turbulence are areas that have engaged medical researchers applying chaotic systems to study cardiovascular physiology and the biophysics of blood flow. In fact, this area has much potential for research in the world of medicine.

FRACTAL IN ASTRONOMY

Fractals may revolutionize the way that the universe is seen. Cosmologists usually assume that matter is spread uniformly across space. But observations shows that this is not true. Astronomers agree with that assumption on small scales, but most of them think that the universe is smooth at very large scales. However, a dissident group of scientists claim that the structure of the universe is fractal at all scales. If this new theory is proved to be correct, even the big bang models should be adapted. Some years ago we proposed a new approach for the analysis of the galaxy and cluster correlations based on the concepts and methods of modern statistical physics. This led to the surprising result that the galaxy correlations are fractal and not homogeneous up to the limits of the available catalogues. The result is that the galaxy structures are highly irregular and self similar.

FRACTAL IN NATURE

Suppose take a tree. Pick a particular branch and study it closely. Choose a bundle of leaves on that branch. To chaologists, all three of the objects described - the tree, the branch and the leaves are identical. To many, the word chaos suggests randomness, unpredictability and perhaps even messiness. Chaos is actually very organised and follows certain patterns. The problem arises in finding these elusive and intricate patterns. One purpose of studying chaos through fractals is to predict patterns in dynamical systems that on the surface seems unpredictable. A system is a set of things, an area of study; a set of equations is a system, as well as more tangible things such as cloud formations, the changing weather, the movement of water currents, or animal migration patterns. Weather forecasts are never totally accurate and long term forecasts, even for a week can be totally wrong. This is due to minor disturbances in airflow, solar heating etc. Each disturbance may be minor, but the change it creates will increase geometrically with time. Soon, the weather will be far different than what was expected. With fractal geometry, we can visually model much of what we witness in nature, the most recognised being coastlines and mountains. Fractals are used to model soil erosion and to analyse seismic patterns as well. Seeing that so many facets of mother nature exhibit fractal properties, may be the whole world around us is a fractal after all.

FRACTAL IN COMPUTER SCIENCE

The most important use of fractals in computer science is the fractal image compression. This kind of compression uses the fact that the real world is well described by fractal geometry. By this way, the images are compressed much more than by usual ways (JPEG or GIF file formats). Another advantage of fractal compression is that when the picture is enlarged, there is no pixelisation. The picture seems very often better when its size is increased.

FRACTAL IN FLUID MECHANICS

The study of turbulence in flows is very adapted to fractals. Turbulent flows are chaotic and very difficult to model correctly. A fractal representation of them helps engineers and physicists to better understand complex flows. Flames can also be simulated. Porous media have a very complex geometry and are well represented by fractal. This is actually used in petroleum science.

FRACTAL IN TELECOMMUNICATION

A new application is fractal shaped antennae that reduce greatly the size and the weight of the antennas. Fractenna is the company which sells these antennas. The benefits depend on the fractal applied, frequency of interest and so on. In general the fractal parts produces fractal loading and makes the antenna smaller for a given frequency of use. Practical shrinkage of 2 - 4 times are realizable for acceptable performance. Surprisingly high performance is attained.

CONCLUSION

While the classical definition of chaos means unpredictability and absence of order, the scientific definition of chaos is based on nonlinear mathematics. Although its principles were already established during the late 19th century by Poincare, they were not mathematically accessible until the work of Lorenz (1963). Today, chaos is defined as ’stochastic behaviour in a deterministic system’, or more colloquially; chaos is apparently lawless behaviour totally ruled by (deterministic) laws.

A new scientific revolution is taking shape from the combination of some new concepts with the enoromous power of computation achieved during the last few years. The resulting new vision of nature has been called ” deterministic chaos”, ”science of complexities” and ” non linear dynamics”. Complex systems can now be studied over time and monitored continually in terms of non linear differential equations.

Through this project, we could gather a lot of information about chaos theory and fractals, especially their relevant applications in human life and nature. We actually covered all the basic ideas related to the topic and some extended application in Medical science. Thus we have arrived at a conclusion that chaos theory as such has great applications in various fields of study and research. Thus it paves a new way for the young researchers and scientists to make wonders in the upcoming years.

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