AN INTRODUCTION TO HYPERGRAPH THEORY

PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR

THE MASTER DEGREE IN MATHEMATICS

BY

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CERTIFICATE

This is to certify that the project entitled " AN INTRODUCTION TO HYPERGRAPH THEORY " is a bonafide record of studies undertaken by ASWATHI S (Reg no. 180011015181), in partial fulfillment of the requirements for the award of M.Sc. Degree in Mathematics at Department of Mathematics, St. Paul's College, Kalamassery, during 2018 – 20.

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DECLARATION

I **ASWATHI S** hereby declare that the project entitled **" An Introduction To Hypergraph Theory"** submitted to department of Mathematics St. Paul's College, Kalamassery in partial requirement for the award of M.Sc Degree in Mathematics, is a work done by me under the guidance and supervision of **Dr.Pramada Ramachandran,** Department of Mathematics, St. Pauls's College , Kalamassery during 2018 – 20.

I also declare that this project has not been previously presented for the award of any other degree, diploma, fellowship, etc.

Date ASWATHI S

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Kalamassery **Aswathi s**

CONTENTS

CHAPTER 1

1.1 AN INTRODUCTION

In mathematics, a hypergraph is a generalization of a graph in which a pair H= (X, E) where X is a set of elements called nodes or vertices, and E is a set of non-empty subsets of X called hyperedges or edges. Therefore, E is a subset of $P(X)\setminus\{\phi\}$,where is the power set of X. The size of vertex set is called the order of the hypergraph, and the size of edges set is the size of the hypergraph.

While graph edges are 2-element subsets of nodes, hyperedges are arbitrary sets of nodes, and can therefore contain an arbitrary number of nodes. However, it is often desirable to study hypergraphs where all hyperedges have the same cardinality. A k-uniform hypergraph is a hypergraph such that all its hyperedges have size k. (In other words, one such hypergraph is a collection of sets, each such set a hyperedge connecting k nodes.) So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. A hypergraph is also called a set system or a family of sets drawn from the universal set.Hypergraphs have many other names. In computational geometry, a hypergraph may sometimes be called a range space and then the hyperedges are called ranges. In cooperative game theory, hypergraphs are called simple games (voting games); this notion is applied to solve problems in social choice theory. In some literature edges are referred to as hyperlinks or connectors.

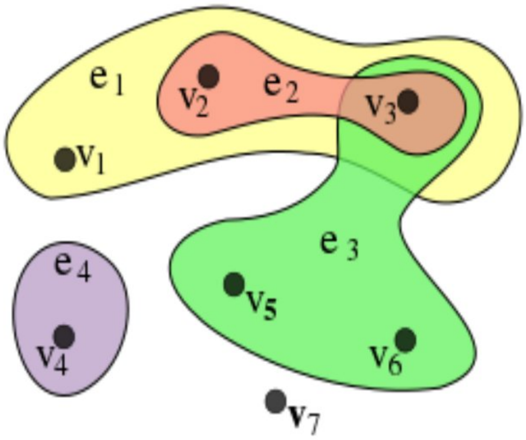
1.2 BASIC CONCEPTS

A hypergraph H denoted by $H = (V;E = (e_i); i\in I)$ on a finite set V is a family $(e_i)_{i\in I}$, (I is a finite set of indexes) of subsets of V called hyperedges.

Sometimes V is denoted by V(H) and E by E(H).

The order of the hypergraph H = (V;E) is the cardinality of V, i.e.

|V| = n; its size is the cardinality of E, i.e. |E| = m. H=(X,E)



An example of a hypergraph, with

$X=\{V_1,V_2,V_3,V_4,V_5,V_6,V_7\}$ and E

$E=\{e_1,e_2,e_3,e_4\}=\{\{V_1,V_2,V_3\},\{V_2,V_3\},\{V_3,V_5,V_6\},\{V_4\}\}$.

This hypergraph has order 7 and size 4. Here, edges do not just connect two vertices but several, and are represented by colors.

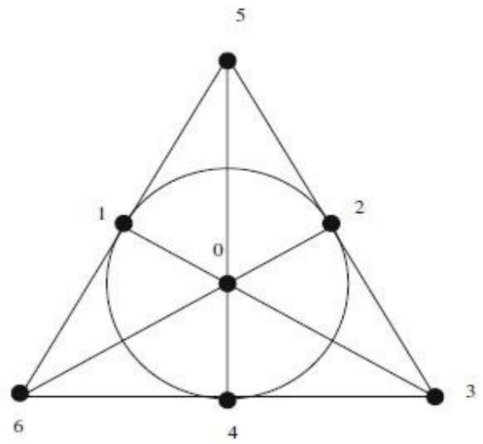
1.3 EXAMPLES OF HYPERGRAPH

Let M be a computer science meeting with $k \geq 1$ sessions : $S_1, S_2, S_3, \ldots, S_k$. Let V be the set of people at this meeting. Assume that each session is attended by one person at least. We can build a hypergraph in the following way:

- The set of vertices is the set of people who attend the meeting;
- the family of hyperedges (e_i) ;$i\in\{1,2,...k\}$ is built in the following way:

– e_i, $i \in \{1, 2, \ldots k\}$ is the subset of people who attend the meeting S_i.

1.3.1 FANO PLANE



The Fano plane is the finite projective plane of order 2, which have the smallest possible number of points and lines, 7 points with 3 points on every line and 3 lines through every point. To a Fano plane we can associate a hypergraph called Fano hypergraph.

- The set of vertices is V = {0, 1, 2, 3, 4, 5, 6};
- The set of hyperedges is E = {013, 045, 026, 124, 346, 235, 156}

The rank is equal to the co-rank which is equal to 3, hence, Fano hypergraph is 3-unifirm. Figure show Fano hypergraph Steiner systems. Let t, k, n be integers which satisfied: $2 \leq t \leq k < n$.

A Steiner system denoted by S(t; k; n) is a k-uniform hypergraph

H = (V;E) with n vertices such that for each subset $T \subseteq V$ with t elements there is exactly one hyperedge $e \in E$ satisfying $T \subseteq e$. For instance the complete graph Kn is a S(2; 2; n) Steiner system. An important example is the Steiner systems S(2; 3; n) which are called Steiner triple systems. The Fano plane is an example of a Steiner triple system on 7 vertices.

1.3.2 LINEAR SPACES

A linear space is a hypergraphs in which each pair of distinct vertices is contained in precisely one edge. To exclude trivial cases, it is always assumed that there are no empty or singleton edge S.

A hypergraph with only one edge which contains all vertices, this is called a trivial linear space.

A simple hypergraph is a hypergraph H = (V; E) such that

$e_i \subseteq e_j \implies i = j$.

A simple hypergraph has no repeated hyperedge.

A hypergraph is linear if it is simple and $|e_i \cap e_j| \leq 1$ for

all $i, j \in I$ where $i = j$.

1.4 GRAPH: PRELIMINARY DEFINITIONS

A graph is an ordered triple $G=(V(G),E(G),I_G)$ where V(G) is a nonempty set, E(G) is a set disjoint from V(G) and I_G is an incidence relation that associates with each element of E(G) an unordered pair of elements (same or distinct) of V(G).Elements of V(G) are called the vertices of G and elements of E(G) are called the edges of G.V(G) and E(G) are the vertex set and edge set of G respectively. If, for the edge e of G ,$I_G(e)=\{u,v\}$,we write $I_G(e)=uv$.

If $I_G(e)=\{u,v\}$, then the vertices u and v are called the end vertices or ends of the edge e.Each edge is said to join its ends; in this case, we say that e is incident with each one of its ends. Also, the vertices u and v are then incident with e: A set of two or more edges of a graph G is called a set of multiple or parallel edges if they have the same pair of distinct ends. If e is an edge with end vertices u and v; we write e= uv: An edge for which the two ends are the same is called a loop at the common vertex. A vertex u is a neighbor of v in G, if uv is an edge of G, and $u \neq v$.The set of all neighbours of v is the open neighborhood of v or the neighbor set of v; and is denoted by N(v); the set $N[v]=N(v) \cup \{v\}$ is the closed neighborhood of v in G: When G needs to be made explicit, these open and closed neighborhoods are denoted by $N_G(v)$; and $N_G[v]$ respectively. Vertices u and v are adjacent to each other in G if and only if there is an edge of G with u and v as its ends. Two distinct edges e and f are said to be adjacent if and only if they have a common end vertex. A graph is simple if it has no loops and no multiple edges. Thus, for a simple graph G, the incidence function I_G is one-to-one. Hence, an edge of a simple graph is identified with the pair of its ends. A simple graph therefore may be considered as an ordered pair (V(G),E(G)), where V(G) is a nonempty set and E(G) is a set of unordered pairs of elements of V(G) (each edge of the graph being identified with the pair of its ends).

A graph is trivial if its vertex set is a singleton and it contain no edges. A graph is bipartite if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y.The pair (X,Y) is called a bipartition of the bipartite graph. The bipartite graph G with bipartition (X,Y) is denoted by G(X,Y). A simple bipartite graph G(X,Y) is complete if each vertex of X is adjacent to all the vertices of Y: If G(X,Y) is complete with $|X|=p$ and $|Y|=q$ then G(X,Y) is denoted by $K_{p,q}$. A complete bipartite graph of the form $K_{1,q}$ is called a star.

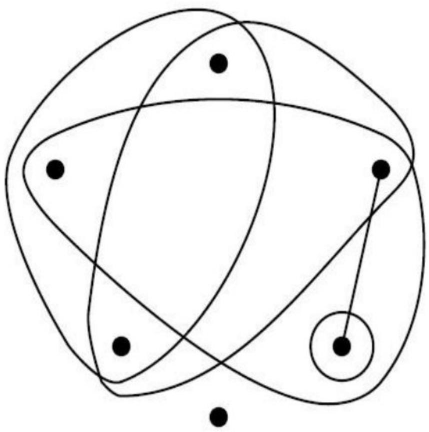
A graph H is called a subgraph of G if $V(H) \subseteq V(G)$,$E(H) \subseteq E(G)$ and I_H is the restriction of I_G to E(H). If H is a subgraph of G,then G is said to be a supergraph of H: A subgraph H of a graph G is a proper subgraph of G if either $V(H) \neq V(G)$ OR $E(H) \neq E(G)$.(Hence, when G is given, for any subgraph H of G, the incidence function is already determined so that H can be specified by its vertex and edge sets.) A subgraph H of G is said to be an induced subgraph of G if each edge of G having its ends in V(H) is also an edge of H. A subgraph H of G is a spanning subgraph of G if V(H)=V(G).

The induced subgraph of G with vertex set S ⊆ V(G)is called the subgraph of G induced by S and is denoted by G[S]. Let E' be a subset of E and let S denote the subset of V consisting of all the end vertices in G of edges in E'. Then the graph (S,E',$I_G \backslash_{E'}$)is the subgraph of G induced by the edge set E' of G. It is denoted by G[E']. Let u and v be vertices of a graph G . By G+uv, we mean the graph obtained by adding a new edge uv to G.

A clique of G is a complete subgraph of G. A clique of G is a maximal clique of G if it is not properly contained in another clique of G.

An automorphism of a graph G is an isomorphism of G onto itself.We recall that two simple graphs G and H are isomorphic if and only if there exists a bijection ϕ:V(G) to V(H) such that uv is an edge of G if and only if ϕ(u) ϕ(v) is an edge of H. In this case ϕ is called an isomorphism of G onto H.

1.5 HYPERGRAPH :PRELIMINARY DEFINITIONS



An example of a hypergraph is shown in above. The basic idea of the hypergraph concept is to consider such a generalization of a graph in which any subset of a given set may be an edge rather than two-element subsets. In drawing hypergraphs, vertices are points in the plane, edges of size 2 are curves connecting respective vertices (as in graph drawing), and edges of size different from 2 are closed curves separating a respective subset from the rest of vertices.

Let $X = \{x_1,x_2, . . . ,x_n\}$ be a finite set, and let $D = \{D_1,D_2, . . . ,D_m\}$ be a family of subsets of X. The pair H = (X,D) is called a hypergraph with vertex set X also denoted by V(H), and with edge set D also denoted by D(H). Sometimes, the hypergraph H = (X,D) is called a set-system. $|X| = n$ is called the order of the hypergraph, written also as n, or n(H). The elements $x_1,x_2, . . . ,x_n$ are called the vertices and the sets $D_1,D_2, . . . ,D_m$ are called the edges (hyperedges). The number of edges is usually denoted by m or m(H). Sometimes we will omit the indices when denoting the vertices and edges if this evidently does not lead to misunderstanding. To include the most general case (it may happen in some algorithms), we assume that the set of vertices X and/or the family D may be empty. A hypergraph which contains no vertices and no edges is called the empty set. Some edges may also be empty sets. Some edges may be the subsets of some other edges; in this case they are called included. In some cases some edges may coincide; they are then called multiple. A hypergraph is called simple if it contains no included edges. Hence simple hypergraphs do not have empty and multiple edges. Simple hypergraphs are also known as Sperner families.

In a hypergraph, two vertices are said to be adjacent if there is an edge $D \in D$ that contains both vertices. The adjacent vertices are sometimes called neighbor to each other, and all the neighbors for a given vertex x are called the neighborhood of x in a graph or hypergraph. The neighborhood of x is denoted by N(x). Two edges are said to be adjacent if their intersection is not empty. If a vertex $x_i \in X$ belongs to an edge $D_j \in D$, then we say that they are incident to each other. As one can see, as in graph theory, the adjacency is referred to the elements of the same kind (vertices vs vertices, or edges vs edges), while the incidence is referred to the elements of different kind (vertices vs edges).

$D(x), x \in X$, will denote all the edges containing the vertex x. The number $|D(x)|$ is called the degree of the vertex x, the number $|D_i|$ is called the degree (size, cardinality) of the edge D_i. The maximum degree of the hypergraph H is denoted by, $r(H) = \max_{x \in X} |D(x)|$. A hypergraph in which all vertices have the same degree $k \geq 0$ is called k-regular .

A hypergraph in which all edges have the same degree $r \geq 0$ is called r-uniform.

The rank of a hypergraph H is $r(H) = \max_{x \in X} |D|$.

An edge of a hypergraph which contains no vertices is called an empty edge. The degree of an empty edge is trivially 0. A vertex of a hypergraph which is incident to no edges is called an isolated vertex. The degree of an isolated vertex is trivially 0. An edge of cardinality 1 is called a singleton (loop), a vertex of degree 1 is called a pendant vertex.

A simple hypergraph H with $|D_i| = 2$ for each $D_i \geq D$ is thus a simple graph, maybe with isolated vertices.

Two simple hypergraphs H_1 and H_2 are called isomorphic if there exists a one-to-one correspondence between their vertex sets such that any subset of vertices form an edge in H_1 if and only if the corresponding subset of vertices forms an edge in H_2.

CHAPTER 2

HYPERGRAPH PROPERTIES

2.1 HYPERGRAPH VARIATION

2.1.1 EMPTY HYPERGRAPH:

By definition the empty hypergraph is the hypergraph such that

$V=\phi$

$E=\phi$

2.1.2 TRIVIAL HYPERGRAPH:

Trivial hypergraph is a hypergraph such that

$V\neq\phi$

$E=\phi$

2.1.3 UNIFORM HYPERGRAPH:

k-uniform hypergaph –When all hyperedges have the same cardinality ;so a 2-uniform hypergraph is a classic graph,a 3-uniform hypergraph is a collection of unordered triples, and so on.

2.1.4 ORDERED HYPERGRAPH :

An r-uniform hypergraph is said to be ordered if the occurence of nodes in every edge is numbered from 1 to r .

2.1.5 SIMPLE HYPERGRAPH: A hypergraph is simple if all edges are distinct.A simple hypergraph is ahypergraph H=(V,E) such that $e_i \subseteq e_j$ implies i=j. A simple hypergraph has no repeated hyperedge.

2.1.6 INDUCED SUB-HYPERGRAPH:

The induced subhypergraph H(V')of the hypergraph H where $V' \subseteq V$ is the hypergraph H(V')=(V',E') defined as ,

$E'=\{V(e_i) \cap V' \neq \phi : e_i \in E$ and either e_i is loop or $|V(e) \cap V'| \geq 2\}$

The letter E' can be represented a multi-set.Moreover ,according to the remark above we can add, if we need the empty set.

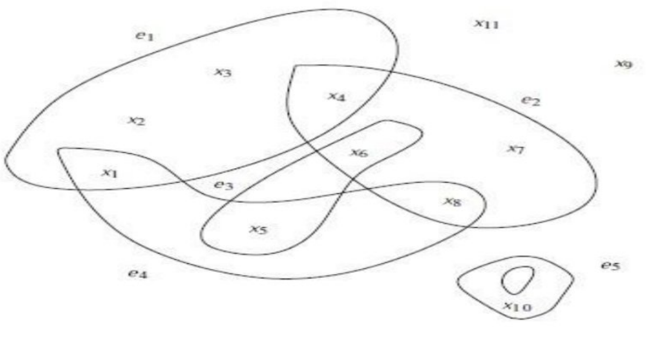
Given a subset $V' \subseteq V$, the subhypergraph,the hypergraph H' is the hypergraph $H'=(V',E'=(e_j)_{j \in J})$such that for all $e_j \in E' : e_j \subseteq V'$.

2.1.7 PARTIAL AND REGULAR HYPERGRAPH:

A partial hypergraph generated by $J \subseteq I$, H'of H is a hypergraph

$H'=(V',(e_j)_{j \in J})$ Where $\cup_{j \in J} e_j \subseteq V'$.Note that we may assume have V'=V.

If each vertex has the Same degree,we say that the hypergraph is regular ,or k-regular if for every $x \in V$,d(x)=k.



2.2 EXAMPLE:

The hypergaraph H has all 11 vertices; 5 hyperedges ;1 loop:e_5 ;

2 isolated vertices: x_{11} ,x_9 .

The rank r(H)= 4, the co-rank cr(H)= 1.

The degree of x is 2.

H'= $(V;\{e_1,e_2\})$ is a partial hypergraph generated by J= {1,2}; $H(V')= (V'=\{x_1, x_4, x_6, x_8, x_{10}\}$;

$e_1'=e_1 \cap v'=\{x_1, x_4\}$;

$e_2'=e_2 \cap v'=\{x_4, x_6, x_8\}$

$e_4'=e_4 \cap v'=\{x_1, x_8\}$

$e_5'=e_5 \cap v'=\{x_{10}\}$ is an induced hypergraph .Notice that $e_3 \cap v'=\{x_6\}$ is not an hyperedge for this induced hypergraph.

$H'= (V'=\{x_1, x_2, x_3, x_4, x_7\}, E=\{e_1\})$ is a sub hypergraph with 1 isolated vertex : x_7 .Hypergraph H is linear and simple.

HYPERGRAPH PROPERTIES:

2.3 GRAPH VERSUS HYPERGRAPHS

Graphs:

A multigraph, Γ = (V; E) is a hypergraph such that the rank of Γ is at most two. The hyperedges are called edges. If the hypergraph is simple, without loop, it is a graph. Consequently any definition for hypergraphs holds for graphs. Given a graph Γ , we denote by Γ (x) the neighborhood of a vertex x, i.e. the set formed by all the vertices which form a edge with x:

$\Gamma(x) = \{y \in V : \{x, y\} \in E\}$

In the same way, we define the neighborhood of $A \subseteq V$ as $\Gamma(A) = \cup_{x\in A} \Gamma(x)$. The open neighborhood of A is $\Gamma^0(A) = \Gamma(A) \setminus A$.

An induced subgraph generated by $V' \subseteq V$ is denoted by $\Gamma(V')$. A graph $\Gamma = (V; E)$ is bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and every edge joins a vertex of V_1 to a vertex V_2.

It is well known that a graph $\Gamma = (V; E)$ is bipartite if and only if it does not contain any cycle with an odd length. A graph is complete if any pair of vertices is an edge.A clique of a graph $\Gamma = (V; E)$ is a complete subgraph of Γ .

The maximal cardinality of a clique of a graph Γ is denoted by $\omega(\Gamma)$.

Remember that a graph is chordal if each of its cycles of four or more vertices has a chord, that is, an edge joining two non-consecutive vertices in the cycle.

GRAPHS AND HYPERGRAPHS

Let $H = (V; E = (e_i)/i \in I)$ be a hypergraph such that $E \neq \emptyset$. The line-graph (or representative graph, but also intersection graph) of H is the graph $L(H) = (V'; E')$ such that:

1. $V' = I$ or $V' := E$ when H is without repeated hyperedge; 2.

$\{i, j\} \in E'$ $(i \neq j)$ if and only if $e_i \cap e_j \neq \emptyset$.

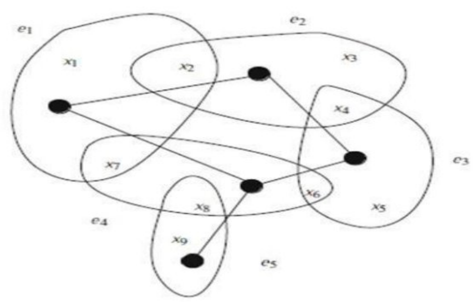


Figure above shows a hypergraph $H = (V; E)$, where

$V = \{x_1, x_2, x_3, \ldots, x_9\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, and its representative. The vertices of L(H) are the black dots and its edges are the curves between these dots.

Some properties of hypergraphs can be seen on the line-graph, for instance it is easy to show that:

2.4 Lemma : The hypergraph H is connected if and only if L(H) is.

2.5 Proposition : Any non trivial graph Γ is the line-graph of a linear hypergraph.

Proof:

Let Γ = (V; E) be a graph with $V = \{x_1, x_2, \dots x_n\}$. Without loosing generality, we suppose that Γ is connected (otherwise we treat the connected components one by one). We can construct a hypergraph H = (W; X) in the following way:

- the set of vertices is the set of edges of Γ , i. e. W = E. It is possible since Γ is simple.
- the collection of hyperedges X is the family of X_i where X_i is the set of edges of Γ having xi as incidence vertex.

So we can write:

$H = (E; X = (X_1, X_2, \dots, X_n))$ with:

$Xi = \{e \in E : x_i \in e\}$ where $i \in \{1, 2, 3, \dots\}$

Notice that if Γ has only one edge then $V = \{x_1, x_2\}$ and $X_1 = X_2$.

It is the only case where H has a repeated hyperedge.

If |E| > 1, if i ≠ j and Xi ∩ X j ≠ ∅; there is exactly one, (since Γ is a simple graph) e ∈ E such that e ∈ Xi ∩ X j with e = {xi , x j }. It is clear that Γ is the line-graph of H .

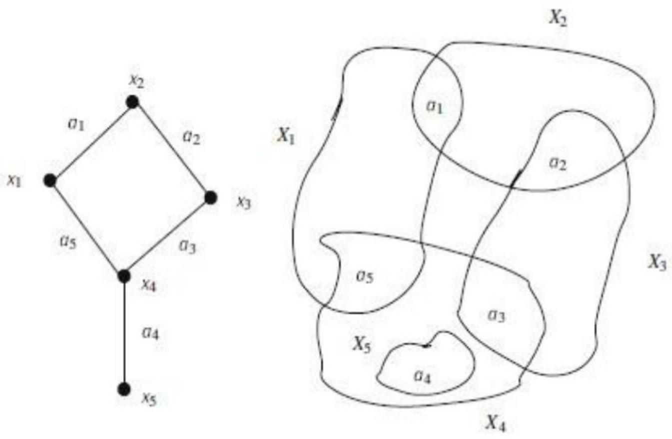


Figure above illustrate proposition .

2.6 : INTERSECTING FAMILIES

Let H = (V; E = $(e_i)_{i\in I}$) be a hypergraph. A subfamily of hyperedges $(e_j)_{j\in J}$, where J ⊆ I is an intersecting family if every pair of hyperedges has a non empty intersection. The maximum cardinality of |J | (of an intersecting family of H) is denoted by $\Delta_0(H)$.

2.7 : HELLY PROPERTY

The Helly property plays a very important role in the theory of hypergraphs as the most important hypergraphs have this property . A hypergraph has the Helly property if each intersecting family has a non-empty intersection (belonging to a star). It is obvious that if a hypergraph contains a triangle it has not the Helly property. A hypergraph having the Helly property will be called Helly hypergraph. A hypergraph has the strong Helly property if each partial induced subhypergraph has the Helly property. The hypergraph shown figure below has the Helly property but it has not the strong Helly property.

THE HELLY PROPERTY:

Let H = (E_1 ,E_2..,. ,E_m)b e a simple hypergraph. We say that H has the Helly property if every intersecting family of H is a star, i.e. for J ⊂ {1,2, ..., m},

$E_j \cap E_k \neq \phi$ (j,k ∈J) implies $\bigcap_{j \in J} E_j$ ϕ.

Hence a graph has the Helly property if and only if its is triangle-free; hypergraphs with the Helly property have also other properties which generalise those of triangle free graphs.

Example 2.7.1. Let H be an interval hypergraph: its vertices are points on a line, and its edges are intervals of points. A theorem of Helly shows that H has the Helly property.

We can characterize the strong Helly property by the following:

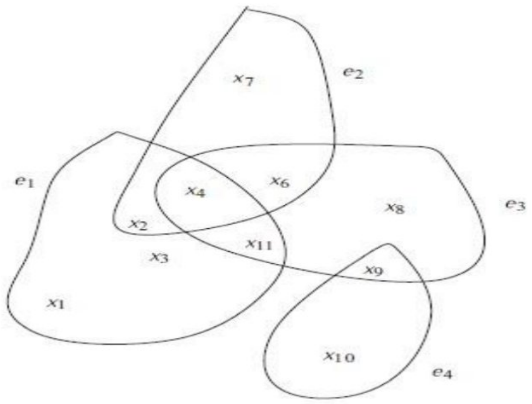
2.8 Theorem :

Let H be a hypergraph. Any partial induced subhypergraph of H has the Helly property if and only if for any three vertices x, y, z and any three hyperedges e_{xy}, e_{xz}, e_{yz} of H, where $x \in e_{xy} \cap e_{xz}$, $y \in e_{xy} \cap e_{yz}$, $z \in e_{xz} \cap e_{yz}$ there exists v ∈ {x, y, z} such that $v \in e_{xy} \cap e_{xz} \cap e_{yz}$.

Proof :

Assume that any partial induced subhypergraph of H has the Helly property. Then, for any three hyperedges e_{xy}, e_{xz}, e_{yz} of H, where $x \in e_{xy} \cap e_{xz}$, $y \in e_{xy} \cap e_{yz}$, $z \in e_{xz} \cap e_{yz}$, just take the partial subhypergraph H(Y) induced by the set Y = {x, y, z} to see that there is a vertex v ∈ {x, y, z} such that: $v \in e_{xy} \cap e_{xz} \cap e_{yz}$.

Fig. The hypergraph above has the Helly property but not the strong Helly property because the induced subhypergraph on Y = V \{x_4} contains the triangle:



$e_1' = e_1 \cap Y$,

$e_2' = e_2 \cap Y$,

$e_3' = e_3 \cap Y$ .We prove the reversed implication by induction on ℓ, the maximal size of an intersecting family of an induced subhypergraph of H. The assertion is clearly true for $\ell = 3$. Assume that for $i = 3, 4, \ldots, \ell$ any partial induced subhypergraph of H with intersecting families of at most ℓ hyperedges has the Helly property.

Let $e_1, e_2, \ldots, e_{l+1}$ be an arbitrary intersecting family of hyperedges of H. By induction,

$\exists x \in \cap_{i\neq 1} e_i$, $\exists y \in \cap_{i\neq 2} e_i$, $\exists z \in \cap_{i\neq 3} e_i$.

As $\{e_1, e_2, e_3\}$ is an intersecting family, there is a vertex $\xi \in \{x, y, z\}$ which is in the intersection $e_1 \cap e_2 \cap e_3$. Hence, $\xi \in \cap_i e_i$ and the assertion holds for $(\ell + 1)$.

2.9 : SUBTREE HYPERGRAPHS

let H = (V; E) be a hypergraph. This hypergraph is called a subtree hypergraph if

• there is a tree Γ with vertex set V such that each hyperedge e ∈ E induces a subtree in Γ .

Conversely, let Γ = (V; A) be a tree, i.e. a connected graph without cycle. We build a connected hypergraph H in the following way:

• the set of vertices of H is the set of vertices of Γ

• the set of hyperedges is a family E = $(ei)_{i\in\{1,2,3....m\}}$ of subset V such that the induced subgraph Γ $(V(e_i))$ is a subtree of Γ , (subgraph which is a tree).

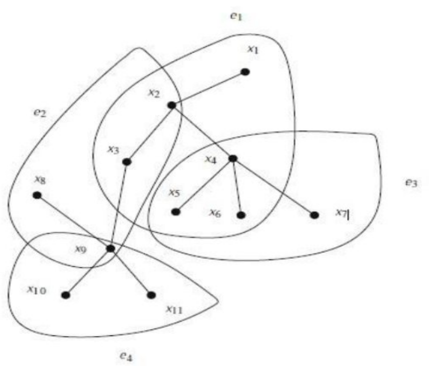


Fig : A subtree hypergraph associated with a tree.

2.10 : STABLE (OR INDEPENDENT), TRANSVERSAL AND MATCHING

Let H = (V; $(ei)_{i\in I}$) be a hypergraph without isolated vertex.

A set A ⊆ V is a stable or an independent (resp. a strong stable) if no hyperedge is contained in A (resp. $|A \cap V(e_i)| \leq 1$, for every i ∈ I).

The stability number α(H) (resp. the strong stability number α'(H)) is the maximum cardinality of a stable (resp. of a strong stable).

A set B ⊆ V is a transversal if it meets every hyperedge i.e. for all $e \in E$, $B \cap V(e) \neq \emptyset$.

The minimum cardinality of a transversal is the transversal number. It is denoted by τ(H).

A matching is a set of pairwise disjoint hyperedges of H.

The matching number ν(H) of H is the maximum cardinality of a matching. A hyperedge cover is a subset of hyperedges:

$(e_j)_{j \in J}$, $(J \subseteq I)$ such that: $\cup_{j \in J} e_j = V$.

The hyperedge covering number, ρ(H) is the minimum cardinality of a hyperedge cover.

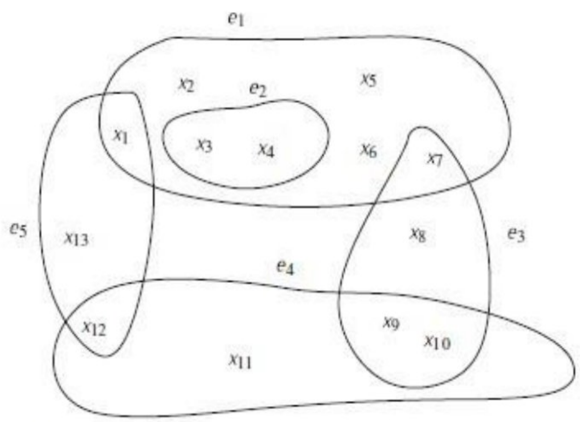


Fig. 2.12 The set $\{x_1; x_3; x_5; x_9; x_{11}; x_{13}\}$ is a stable of the hypergraph above but it is not a strong stable. The set $\{x_3; x_8; x_{11}; x_{13}\}$ is a transversal; $\tau(H) = 3$, $\rho(H) = 4$ and $\nu(H) = 3$. It is conformal and it has the Helly property

2.10.1 EXAMPLES :

(1) The problem of scheduling the presentations in a conference is an example of the maximum independent set problem. Let us suppose that people are going to present their works, where each work may have more than one author and each person may have more than one work.

The goal is to assign as many presentations as possible to the same time slot under the condition that each person can present at most one work in the same time slot.

We construct a hypergraph with a vertex for each work and a hyperedge for each person, it is the set of works that he (or she) presents. Then a maximum strong independent set represents the maximum number of presentations that can be given at the same time.

(2) The problem of hiring a set of engineers at a factory is an example of the minimum transversal set problem.

Let us suppose that engineers apply for positions with the lists of proficiency they may have, the factory management then tries to hire the least possible number of engineers so that each proficiency that the factory needs is covered by at least one engineer.

We construct a hypergraph with a vertex for each engineer and an hyperedge for each proficiency, then a minimum transversal set represents the minimum group of engineers that need to be hired to cover all proficiencies at this factory.

2.11 :KONIG PROPERTY : A matching in a hypergraph H is a family of pairwise disjoint edges, and the maximum cardinality of a matching is denoted $\nu(H)$.

A matching can also be defined as a partial hypergraph H_0 with $\Delta(H_0) = 1$.

We note that for every transversal T and for every matching H_0,

$|T \cap E| \geq 1 \quad (E \in H_0)$ Thus

$|H_0| \leq |T|$, from when

$\nu(H) = \max |H_0| \leq \tau(H)$.

We say that H has the Konig property if $\nu(H) = \tau(H)$.

A covering of H will be a family of edges which covers all the vertices of H , that is to say a partial hypergraph H_1 with $\delta(H_1) = \min_{x\in X} d_{H1}(x) \geq 1$.

We write $p(H) = \min|H_1|$.

Finally, a strongly stable set of H is by definition a set $S \subset X$ such that $|S \cap E_1| \leq 1$ for every $E \in H$, and we write $\overline{\alpha}(H) = \max |S|$.

2.12 :TRANSVERSAL HYPERGRAPHS

Let $H = (E_1, .. ., E_m)$ be a hypergraph on a set X . A set $T \subset X$ is a transversal of H if it meets all the edges, that is to say:

$T \cap E_i = \phi \quad (i = 1,2, ...,m)$

The family of minimal transversals of H constitutes a simple hypergraph on X called the transversal hypergraph of H , and denoted by Tr H.

Example 2.12.1. If the hypergraph is a simple graph G, a set S is stable if it contains no edge, that is, if its complement X-S meets all the edges of G. Thus, Tr G = {X-S/S is a maximal stable set of G}.

Example 2.12.2. The complete r-uniform hypergraph K^r_n on X admits as minimal transversals all the subsets of X with n-r+1 elements. Thus $Tr\left(k\,{}^{r}_{n}\right) = .k\,{}^{n-r+1}_{n}$.

2.13 The coefficients τ and τ':

For a hypergraph H we denote by $\tau(H)$ the transversal number, that is to say, the smallest cardinality of a transversal; similarly, we denote by $\tau'(H)$ the largest cardinality of a minimal transversal. Clearly:

$$\tau(H) = \min_{T \in TrH} |T| \leq \max_{T \in TrH} |T| = \tau'(H)$$

Example 2.13.1: The finite projective plane of r: By definition, a projective plane of rank r is a hypergraph having r^2 - r + 1 vertices ("points"), and r^2 - r + 1 edges ("lines"), satisfying the following axioms:

(1) every point belongs to exactly r lines;

(2) every line contains exactly r points;

(3) two distinct points are on one and only one line;

(4) two distinct lines have exactly one point in common.

Projective planes do not exist for every value of r (for example, if r = 7), but it is known that if r = p^{α} + 1, with p prime, p ≥ 2,α ≥1, there exists a projective plane of rank r denoted PG(2, p^{α}) built on a field of p^{α} elements. For example, the projective plane with seven points ("Fano configuration") is PG(2,2). It is clear that in a projective plane every line is a minimal transversal set of H .

In the projective plane of seven points there are no others because H = Tr H (given that any two edges meet and that the chromatic number of this hypergraph is > 2) .For the projective planes of rank r > 3, we have τ(H) = r , but there exist other minimal transversals which are all of cardinality ≥ r + 2 .
Hence τ '(H) ≥ r + 2.

2.14 τ-critical hypergraphs:

We say that a hypergraph H = (E_1,E_2..,. ,E_m) is τ-critical if the deletion of any edge decreases the transversal number, that is to say, if

$$\tau(H-E_j) < \tau (H) \quad (j = 1,2, ..., m)$$

Since we cannot have τ (H-E_j)< τ(H) - 1 , this is equivalent to saying that if H is τ-critical with τ (H) = t+1, then τ(H-E) = t for every E$\in$H .

2.14.1 Example:

The hypergraph k_{t+r}^{r} is τ-critical, since $\tau(k_{t+r}^{r})$= t + 1 and if E is an edge of $k_{t+r}^{r}$the hypergraph k $_{t+r}^{r}$- E has a transversal X-E of cardinality t .

2.15.1 Proposition :

Every τ-critical hypergraph is simple.

Proof : For if H = (E_1.,. .,E_m) is τ-critical and not simple, there exist two indices i and j with $E_i \subset E_j$. An optimal transversal of H-E_j has τ(H)-1 vertices, and since it meets E_i it also meets E_j. Therefore τ(H)≤ τ(H)-1, a contradiction.

2.15.2 Proposition :

Every hypergraph H with τ(H) = t + 1 has as a partial hypergraph, a τ-critical hypergraph H' with τ(H') = t+1.

Indeed, to obtain H' it is enough to remove from H as many edges as one can without changing the transversal number.

In a hypergraph H a vertex x is said to be critical if

τ (H-H(x)) < τ (H) -------- (1)

We note that (1) is equivalent to:

τ(H-H(x)) = τ(H)-1 ---------(2)

proof:

Indeed, if (1) holds then the hypergraph H_1 = H - H (x) has a transversal T_1 of cardinality τ(H)-1. The set $T_1 \cup$ {x} is a transversal of H and, since its cardinality is τ(H) , it is a minimum transversal. From this we obtain (2).

Conversely, if (2) holds, let T be a minimum transversal of H containing x . Then

T-{x} is a transversal of H-H(x) of cardinality τ(H)-1, from which (1) follows.

2.15.3 Proposition :

Every vertex of a τ-critical hypergraph is critical.

Proof:

Let H be a τ-critical hypergraph and let x be one of its vertices. Since x is contained in an edge, E say, $\tau(H - H (x)) \leq \tau(H-E) < \tau(H)$.

Thus x is a critical vertex.

Example .

Let us consider a simple graph G = (X ,E), connected and without bridges. Let H be the hypergraph whose vertices are the edges of G and whose edges are the elementary cycles of G. Through every edge of a graph without bridges there passes a cycle. Hence H is a simple hypergraph on E .

For $e_0 \in$ E there exists a maximal tree (X,F) with $e_0 \in$ F which spans G ; we have $\tau(H) = m(G)- n(G)+1$, and every co-tree of G is a transversal of H . Therefore E-F is a minimum transversal of H containing e_0. Thus every vertex of H is critical.

CHAPTER 3

HYPERGRAPH COLORINGS

3.1 :Coloring

Let $H = (V; E = (e_i)_{i \in I})$be a hypergraph and $k \geq 2$ be an integer.

A k coloring of the vertices of H is an allocation of colors to the vertices such that:

(i) A vertex has just one color.

(ii) We use k colors to color the vertices.

(iii) No hyperedge with a cardinality more than 1 is monochromatic.

From this definition it is easy to see that any coloring induces a partition of the set of vertices in k classes:

$(C_1, C_2, C_3, \ldots, C_k)$ such that for $e \in E(H)$, $|e| > 1$ then e C_i ,

$\forall i \in \{1, 2, 3, \ldots, k\}$.Then e subset or equal to C_1 , for all $i \in \{1,2,\ldots.k\}$.

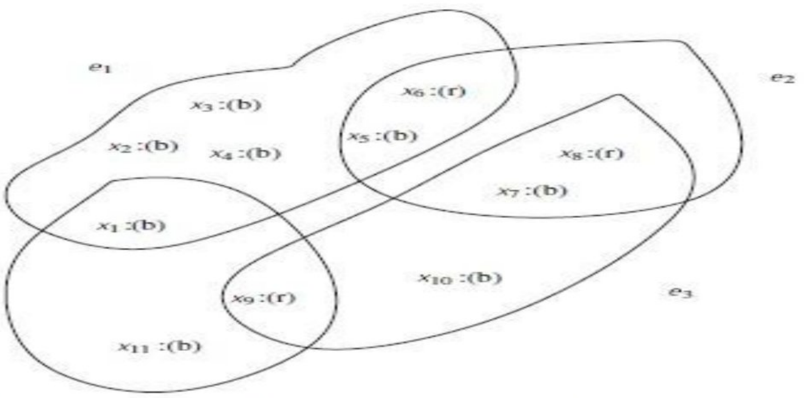


Figure shows a colored hypergraph H where (r) is *red* and (b) is *blue*. We have $\chi(H) = 2$

The chromatic number $\chi(H)$ of H is the smallest k such that H has a k-coloring.

3.1.1 EXAMPLE:

If H is the hypergraph which vertices are the different waste products of a chemical production factory, and which hyperedges are the dangerous combinations of these waste products. The chromatic number of H is the smallest number of waste disposal sites that the factory needs in order to avoid any dangerous situation.

3.2 : Chromatic Number

Let $H = (E_1, E_2, \ldots, E_n)$ be a hypergraph and let k be an integer ≥ 2 .

A k-coloring (of the vertices) is a partition $(S_1, S_2 .., . , S_k)$ of the set of vertices into k classes such that every edge which is not a loop meets at least two classes of the partition; that is to say

$$E \in H, |E| > 1 \Longrightarrow E \not\subset S_i \ (i = 1, 2, \ldots, k).$$

A vertex in Si will be said to be a "vertex of colour i", and S_i ("the colour set i") may possibly be empty; the only "monochromatic" edges are therefore the loops. For a hypergraph H its chromatic number $\chi(H)$ is the smallest integer k for which H admits a k-colouring.

3.2.1 Example:

If H is the hypergraph whose vertices are the different waste products in a chemical production factory, and in which the edges are the dangerous combinations of these waste products, the chromatic number of H is the smallest number of waste disposal sites that the factory needs in order to avoid any hazardous situation.

We note that if the hypergraph H is a graph, the chromatic number of H coincides exactly with the usual chromatic number.

For a hypergraph H on X , a set $S \subset X$ is said to be stable if it does not contain any edge E with $|E| > 1$. The stability number $\alpha(H)$ o f H is the maximum cardinality of a stable set of H.

3.3 :PARTICULAR COLORINGS

3.3.1 STRONG COLORING:

Let H = (V; E) be a hypergraph, a strong k-coloring is a partition (C_1, C_2, . . . , C_k) of V such that the same color does not appear twice in the same hyperedge. In another words:

$|e \cap C_i| \leq 1$ for any hyperedge and any element of the partition.

The strong chromatic number denoted by χ(H) is the smallest k such that H has a strong k-coloring.

3.3.2 EQUITABLE COLORING

Let H = (V; E) be a hypergraph, an equitable k-coloring is a k-partition (C_1,C_2, . . . ,C_k) of V such that, in every hyperedge e, all the colors {1, 2, . . . , k} appear the same number of times, to within one, if k does not divide $|e_i|$.

It is:

$$for\ all\ e \in\ E\ , \left\lfloor\frac{|e|}{k}\right\rfloor \leq |e \cap C_i| \leq \left\lceil\frac{|e|}{k}\right\rceil\ , i \in \{1,2,\dots,k\}$$

It is easy to see that a strong k-coloring is an equitable k-coloring.

3.3.3 GOOD COLORING:

Let H = (V; E) be a hypergraph, a good k-coloring is a k-partition $(C_1, C_2, \ldots, C_K)$ of V such that every hyperedge e contains the largest possible number of different colors, i.e. for every $e \in E$, the number of colors in e is $\min\{|e|; k\}$.

3.3.3.1 Example:

Suppose a network for mobile phones. We can model this network by a hypergraph in the following way:

- the set of vertices is the set of transmission relays.
- a hyperedge is a set of transmission relays which can pairwise interfere and maximal for this property.

If we model a frequency by a color, a good coloring gives us the minimal number of frequencies, k, we need so that communications do not interfere. In that case we have necessarily $k \geq r(H)$, (r (H) is the rank of H).

3.4 Lemma:

Let H = (V; E) be a hypergraph (with $m = |E|$), and $C = (C_1, C_2, \ldots, C_k)$ be a good k-coloring of H, we have:

(i) if $k \leq cr(H)$, (cr(H) is the co-rank of H) then C is a partition in k transversal sets;

(ii) if $k \geq r(H)$ then the good coloring C is a strong coloring.

Proof : Assume that $k \leq cr(H)$.

By definition of a good coloring, if C_i is a set of vertices with color i, we must have:

$C_i \cap e_i \neq \emptyset, \forall j \in \{1, 2, \ldots, m\}$.

Hence C_i is a transversal of H. Assume now that $k \geq r(H)$. Let $e \in E$, then $k \geq |e|$, any two vertices belonging to e have different colors. Consequently, by definition of a strong coloring, the good coloring C is a strong coloring.

3.5 UNIFORM COLORING:

Let H = (V; E) be a hypergraph with $|V| = n$.

A uniform k-coloring is a k-partition:

$(C_1, C_2, \ldots, C_K)$ of V such that the number of vertices of the same color is always the same, to within one, if k does not divide n, i.e.

$$\left\lfloor \frac{n}{k} \right\rfloor \leq |C_i| \leq \left\lceil \frac{n}{k} \right\rceil, i \in \{1,2, \ldots, k\}$$

3.5.1 Example: A airplane manufacturer has p days to construct a plane. If it exceeds these p days, it pays a fine for each extra day. The construction of the plane can be decomposed into n tasks:

$V = \{x_1, x_2, x_3, \ldots, x_n\}$.

One task can be done in a day and a task is made by a workshop. Some employees can make a set of tasks:

$e_1 \subseteq \{x_1, x_2, x_3, \ldots, x_n\}$, some others

can make a set of tasks:

$e_2 \subseteq \{x_1, x_2, x_3, \ldots, x_n\}$ and

so on with $\cup_i e_i = V$.

So we have a hypergraph on V without isolated vertex.

3.6 HYPEREDGE COLORING

Let $H = (V; E)$ be a hypergraph, a hyperedge k-coloring of H is a coloring of the hyperedges such that:

(i) A hyperedge has just one color.

(ii) We use k colors to color the hyperedges.

(iii) Two distinct intersecting hyperedges receive two different colors.

The size of a minimum hyperedge k-coloring is the chromatic index of H. We will denote it by $q(H)$.

3.7 LEMMA

Let H be a hypergraph.

We have: $q(H) \geq \Delta_0(H) \geq \Delta(H)$.

Where $\Delta_0(H)$ is the maximum cardinality of the intersecting families and $\Delta(H)$ is maximum cardinality of the stars.

Proof :

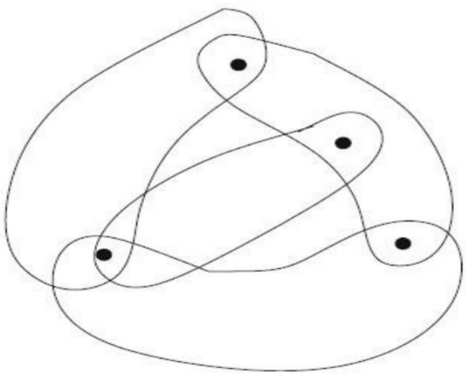
Assume that $\Delta_0(H) = l$. We need l distinct colors to color an intersecting family with at least l hyperedges. Hence $q(H) \geq \Delta_0(H) \geq \Delta(H)$.

A hypergraph has the hyperedge coloring property if $q(H) = \Delta(H)$. For instance a star has the hyperedge coloring property.

3.8 BICOLORABLE HYPERGRAPHS

Bicolorable (or 2-colorable) hypergraphs are a generalization of bipartite graphs.We remind the reader that a graph is bipartite if and only if it is bicolorable. Recognizing if a graph is bipartite can be done in polynomial time.

This is not the case for bicolorable hypergraphs: the problem of recognizing bicolorable hypergraphs is well known to be



Hypergraph which is not bicolorable.

Sometime bicolorable hypergraphs are called bipartite hypergraphs.

Figure above shows a non bicolorable hypergraph.

A cycle $(x_1, e_1, x_2, e_2, \ldots, x_k, e_k, x_1)$ is odd if it has a odd number of hyperedges.

An odd cycle $(x_1, e_1, x_2, e_2, \ldots, x_k, e_k, x_1)$ with distinct vertices and

$X_1 \in e_1 \cap e_k$ is a Sterboul cycle if two non consecutive hyperedges are disjoint and, for every $i = 1, 2, \ldots, k-1$, $|e_i \cap e_{i+1}| = 1$.

3.9 THEOREM: Let $\Gamma = (V; A)$ be a tree and $H = (V; E)$ is a subtree hypergraph associated with Γ, then H is bicolorable.

Proof:

If Γ is a tree, it is bicolorable and any subtree has a induced bicoloring from the 2-coloring of Γ. So H has a bicoloring.

A hypergraph H = (V; E) is critical if it is not 2-colorable but any proper subhypergraph is 2-colorable.

For instance, Fano hypergraph is critical.

3.10 GOOD EDGE COLORING

Let k be an integer ≥ 2. A weak k-coloring of the edges of a hypergraph H is the coloring defined by a weak k-coloring of the dual hypergraph H*. It is thus a partition $H = H_1+H_2+ ...+ H_k$ (edge-disjoint sum) such that for every vertex x with d_H (x) > 1, the star H (x) has at least two edges of different colors. A good k-colouring of the edges of H is a weak k-colouring of the edges of H such that if d_H (x)≥ k,the star H (z) contains at least one edge of each of the colours, and if d_H (x)≤k ,the edges of the star H(z) all have different colours. A strong kcolouring of the edges of H is a partition $H = H_1+H_2+ ...+ H_k$ such that the edges of the star H (z) all have different colours. The chromatic index of H is the smallest value of k for which a strong k-colouring of the edges exists; it is thus the strong chromatic number {H*).

3.11 DEFINITION: Given a hypergraph H , we call a "positional game on H" the situation where two players, say A and B , play in turn at colouring a vertex of H , with the colour red for A and the colour blue for B. A vertex already coloured cannot be recoloured; the winner is the one who first colours an edge of H completely with his colour. If neither of the players obtains a monochromatic edge then the game is a draw.

<u>Example 1</u>. Tic-Tac-Toe in p dimensions.

This is played on the set of cells of a hypercube of p dimensions of sides equal to r , considered as a hypergraph on r^p vertices (the cells of the hypercube) in which the edges are all the sets of r cells that are in line. This game has been studied by Hales and Jewett [1963], who showed that if r is odd and 3^p -1 or r is even and $\geq 2^{p+1}$-2, then player B can force a draw.

One can also play by trying to colour three points in a line with the same colour on any configuration at all, for example the projective plane with seven points.

Example 2.

Ramsey games. Two players A and B play alternately colouring respectively in red and blue an edge of the complete graph K_n , on vertices; the first player to colour with his colour all the edges of a k-clique has won, and his opponent has lost.The hypergraph H_n, which must be considered has $\binom{n}{2}$ vertices and is $\binom{k}{2}$-uniform. A celebrated theory of Ramsey states that there exists an integer R (k,k) such that for every n≥ R(k,k), the hypergraph H_n, has no bicolouring (so that, in consequence, the first player has a winning strategy); if n (k) denotes the smallest order for which the first player wins, we have n(k) ≤ R (k,k) .

3.12 :FUNDAMENTAL PROPOSITION.

In a positional game on a hypergraph H which admits no uniform bicolouring, the first player A has a strategy which assures him a win.

Proof: If H does not have a uniform bicolouring, there necessarily exists a monochromatic edge when all the vertices have been coloured. Thus it is not possible to have a drawn game. This implies, by the theorem of Zermelc-von Neumann, that either player A or player B has a winning strategy.

We argue by contradiction, and suppose that it is the second player B who has a winning strategy σ . Thus, with the following sequence of moves:

x_1, $y_1 = \sigma(x_1)$, x_2,y_2=$\sigma(x_1, x_2)$, x_3 ,y_3 =$\sigma(x_1, x_2, x_3)$,etc.

the first monochromatic edge will be blue, B's colour. However the first player A can play according to the following rule: x_0 being an arbitrary vertex, A's first choice will be $x_1 = \sigma(x_0)$; A's second choice will be $x_2 = \sigma(x_0, y_1)$; etc. (If at any step, $y_i = x_0$, that is to say player B chooses the arbitrary vertex x_0, the player A will play in the same manner with $x_{i+1} = \sigma(x_0, y_1, y_2..., y_i')$ where y_i' is a new arbitrary vertex not already coloured). In this manner A is assured of obtaining a win, and the first monochromatic edge will be red: a contradiction.

CHAPTER 4

APPLICATIONS OF HYPERGRAPH THEORY AND GENERALIZATIONS

4.1 :Hypergraph Theory and System Modeling for Engineering

Modeling is a particularly important aspect in apprehending the continuous or discrete physical systems. The mathematical foundations of the modeling come from:

- Algebraic theory
- The concepts of duality
- Complex and real analysis
- And many others

Since combinatorics is the common denominator of these mathematical areas, combinatorial paradigms are suited to express the mathematical properties of physical objects. Thus, it is natural to develop the hypergraph theory as a modeling concept.

In this section, we are going to briefly present some applications of hypergraphs in science and engineering. It turns out that hypergraph theory can be used in many areas of sciences.

4.2 :Chemical Hypergraph Theory

The graph theory is very useful in chemistry. The representation of molecular structures by graphs is widely used in computational chemistry. But the main drawback of the graph theory is the lack of convenient tools to represent organometallic compounds, benzenoid systems and so on. A hypergraph H = (V, E) is a molecular hypergraph if it represents molecular structure, where $x \in V$ corresponds to an individual atom, hyperedges with degrees greater than 2 correspond to polycentric bonds and hyperedges with deg(x) = 2 correspond to simple covalent bonds.Hypergraphs appear to be more convenient to describe some chemical structures.Hence the concept of molecular hypergraph may be seen as generalization of the concept of molecular graph.

4.3 :Hypergraph Theory for Telecommunications

A hypergraph theory can be used to model cellular mobile communication systems.A cellular system is a set of cells where two cells can use the same channel if the distance between them is at least some predefined value D. This situation can be represented by a graph where:

(a) Each vertex represents a cell.

(b) An edge exists between two vertices if and only if the distance between the corresponding cells is less than the distance called reuse distance and denoted by D.

A forbidden set is a group of cells all of which cannot use a channel simultaneously.

A minimal forbidden set is a forbidden set which is minimal with respect to this property, i.e. no proper subset of a minimal forbidden set is forbidden.

From these definitions it is possible to derive a better modelization using hypergraphs.

We proceed in the following way:

(a) Each vertex represents a cell.

(b) A hyperedge is minimal forbidden set.

4.4 :Hypergraph Theory and Parallel Data Structures

Hypergraphs provide an effective mean of modeling parallel data structures.A shared memory multiprocessor system consists of a number of processors and memory modules. We define a template as a set of data elements that need to be processed in parallel. Hence the data elements from a template should be stored in different memory modules. So we define a hypergraph in the following way:

(a) A data is represented by a vertex.

(b) The hyperedges are the templates.

4.5 :Hypergraphs and Constraint Satisfaction Problems

A constraint satisfaction problem, P is defined as a tuple: P

= $(V, D, R_1(S_1), \ldots, R_k(S_k))$

where:

• V is a finite set of variables.

• D is a finite set of values which is called the domain of P.

• Each Ri (Si) is a constraint.

– S_i is an ordered list of n_i variables, called the constraint scope.

– R_i is a relation over D of arity n_i , called the constraint relation.

To a constraint satisfaction problem one can associate a hypergraph in the following way:

(a) The vertices of the hypergraph are the variables of the problem.

(b) There is a hyperedge containing the vertices $v_1, v_2, ..., v_t$ when there is some constraint R_i (S_i) with scope S = {$v_1, v_2, . . . v_t$ }.

Hypergraph theory can lead to numerous other applications . Indeed we can find hypergraph models in machine learning,data mining, and so on.The properties of hypergraphs are equally important.

for example; hypergraph transversal computation has a large number of applications in many areas of computer science, such as distribued systems, databases, artificial intelligence, and so on. Hypergraph partitioning is also a very interesting property. The partitioning of a hypergraph can be defined as follows:

(a) The set of vertices is partioned into k disjoint subsets $V_1, V_2, ..., V_k$ (b) The partial subhypergraphs (or the set of hyperedges) generated by $V_1, V_2, ..., V_k$ verify the properties $P_1, P_2, ..., P_k$.

This property yields interesting results in many areas such as VLSI design, data mining, and so on.

Directed hypergraphs can be very useful in many areas of sciences. Indeed directed hypergraphs are used as models in:

• Formal languages.

• Relational data bases.

• Scheduling.

and many other applications. Numerous computational studies using hypergraphs have shown the importance of this field in many areas of science and other fruitful applications should be developed in the future.

4.6 Generalizations

One possible generalization of a hypergraph is to allow edges to point at other edges. There are two variations of this generalization. In one, the edges consist not only of a set of vertices, but may also contain subsets of vertices, subsets of subsets of vertices and so on ad infinitum. In essence, every edge is just an internal node of a tree or directed acyclic graph, and vertices are the leaf nodes. A hypergraph is then just a collection of trees with common, shared nodes (that is, a given internal node or leaf may occur in several different trees). Conversely, every collection of trees can be understood as this generalized hypergraph. Since trees are widely used throughout computer science and many other branches of mathematics, one could say that hypergraphs appear naturally as well. So, for example, this generalization arises naturally as a model of term algebra; edges correspond to terms and vertices correspond to constants or variables. For such a hypergraph, set membership then provides an ordering, but the ordering is neither a partial order nor a preorder, since it is not transitive. The graph corresponding to the Levi graph of this generalization is a directed acyclic graph. Consider, for example, the generalized hypergraph whose vertex set is V={a,b} and whose edges are e_1={a,b}{ and e_2={a,e_1 }. Then although b € e_1 and e_1 € e_2 ,it is not true that b € e_2. However, the transitive closure of set membership for such hypergraphs does induce a partial order, and "flattens" the hypergraph into a partially ordered set.

Alternately, edges can be allowed to point at other edges, irrespective of the requirement that the edges be ordered as directed, acyclic graphs. This allows graphs with edge-loops, which need not contain vertices at all. For example, consider the generalized hypergraph consisting of two edges e_1 and e_2 and zero vertices ,so that $e_1 = \{e_2\}$ and $e_2 = \{e_1\}$.As this loop is infinitely recursive, sets that are the edges violate the axiom of foundation. In particular, there is no transitive closure of set membership for such hypergraphs. Although such structures may seem strange at first, they can be readily understood by noting that the equivalent generalization of their Levi graph is no longer bipartite, but is rather just some general directed graph.

The generalized incidence matrix for such hypergraphs is, by definition, a square matrix, of a rank equal to the total number of vertices plus edges. Thus, for the above example, the incidence matrix is simply

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

CONCLUSION

In recent decades ,the theory of hypergraphs has been applied to real life problems.The tools of hypergraph theory can be used for modelling networks,biological networks,data structures,scheduling processes and computations,and many other systems with complex relationships between the entities.From the theoretical point of view ,hypergraphs make it possible to generalize certain theorems in graph theory or even replace a number of theorems on graphs by one theorem on hypergraphs .However ,the majority of the potentials in the development of hypergraph theory are blocked due to inconsistencies in the basic terms.It is proposed to make up a list of basic terms related to hypergraph theory,which can help standardize the graph.

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