<u>PLANAR GRAPHS</u>

PROJECT SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR

THE **BACHELOR OF SCIENCE DEGREE IN MATHEMATICS 2017-2020**

BY

ANJALI M

Reg No. 170021032397

MUHAZINA HAMEED

Reg No. 170021032422

SHILPA VARGHESE

Reg No. 170021032432

UNDER THE GUIDANCE OF

Mr. SANEESH KUMAR V.G

DEPARTMENT OF MATHEMATICS

ST. PAUL'S COLLEGE, KALAMASSERY

(AFFILIATED TO M.G. UNIVERSITY, KOTTAYAM)

2017-2020

CERTIFICATE

This is to certify that the project report entitled "PLANAR GRAPHS" is a bonafide record of studies undertaken by ANJALI M (Reg no. 170021032397),MUHAZINAHAMEED (Reg No. 170021032422)**,** SHILPA VARGHESE (Reg No. 170021032432) in partial fulfilment of the requirements for the award of B.Sc. Degree in Mathematics at Department of Mathematics, St. Paul's College, Kalamassery, during the academic year 2017-2020.

Mr. SANEESH KUMAR V.G	**Dr. SAVITHA K.S**
Project supervisor	**Head of the Department**
Guest Faculty	**Assistant Professor**
Department of Mathematics	**Department of Mathematics**

Examiner

DECLARATION

We, ANJALI M (Reg no. 170021032397), MUHAZINA HAMEED (Reg No. 170021032422)**,** SHILPA VARGHESE (Reg No. 170021032432) hereby declare that this project entitled "PLANAR GRAPHS" submitted to Department of Mathematics of St. Paul's college, Kalamassery in partial requirement for the award of B.Sc Degree in Mathematics, is a work done by us under the guidance and supervision of Mr. SANEESH KUMAR V.G, Department of Mathematics, St. Paul's college, Kalamassery during the academic year 2017-2020

We also declare that this project has not been previously presented for the award of any other degree, diploma, fellowship, etc.

KALAMASSERY ANJALI M

MUHAZINA HAMEED

SHILPA VARGHESE

ACKNOWLEDGEMENT

We express our heartfelt gratitude to our project supervisor Mr. SANEESH KUMAR V.G, Department of Mathematics, for providing us necessary stimulus for the preparation of this project.

We would like to acknowledge our deep sense of gratitude to Dr. SAVITHA K.S, Head of the Department of Mathematics and all other teachers of the department and classmates for their help at all stages.

We also express our sincere gratitude to Ms. VALENTINE D'CRUZ, Principal, St. Paul's College, Kalamassery for the support and inspiration rendered to us in this project report.

KALAMASSERY ANJALI M

MUHAZINA HAMEED

SHILPA VARGHESE

CONTENTS

CHAPTER 1

INTRODUCTION

The development of the graph theory is very much similar to the development of the probability theory. The original work of the graph theory was motivated by constant efforts to understand or solve real life problems. It is no coincidence that different mathematicians have been discovering the graphs theory many times independently. Graph theory is very important area of applied mathematics.

In 1936,the psychologist Lewin used planar graphs to represent the life space of an individual. Kuratowski discovered several other criteria for the identification of planarity of graphs. Tutte developed an important algorithm for drawing a planar graph in a plane.

In graph theory, a planar graph is a graph that can be embedded in the plane ,i.e, it can be drawn on the plane in such a way that its edges intersect only at their end points. In other words, it can be drawn in such a way that no edges cross each other.

CHAPTER -2

PLANAR GRAPHS

2.1-PLANE AND PLANAR GRAPH

Definition:

A plane graph is a graph drawn in the plane in such a way that any pair of edges meet only at their end vertices (if they meet at all)

A planar graph is a graph which is isomorphic to a plane graph i.e, it can be (re)drawn as a plane graph

A graph that cannot be drawn on a plane without a crossover between its edges is called non-planar.

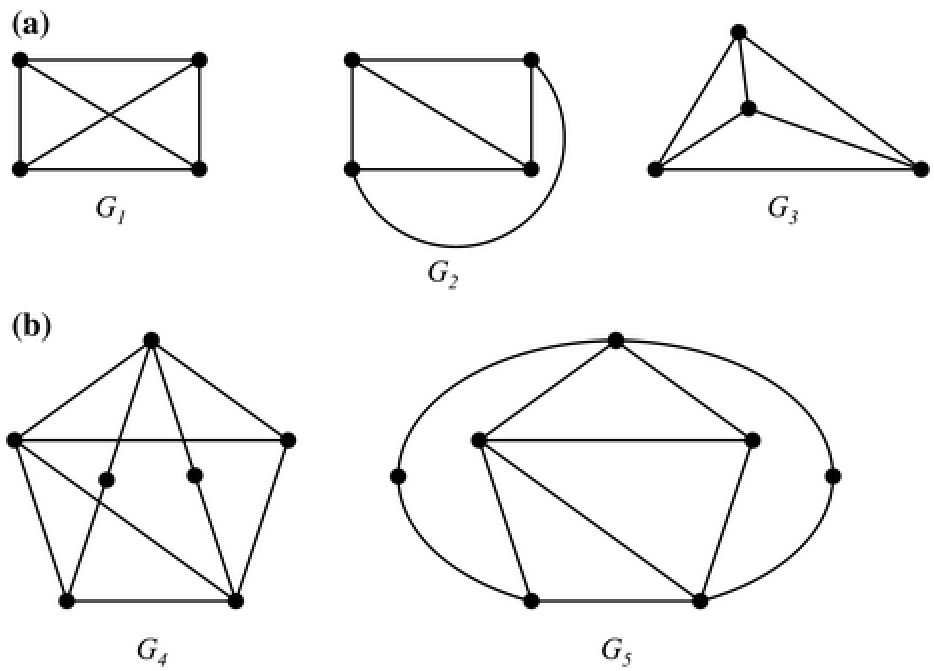
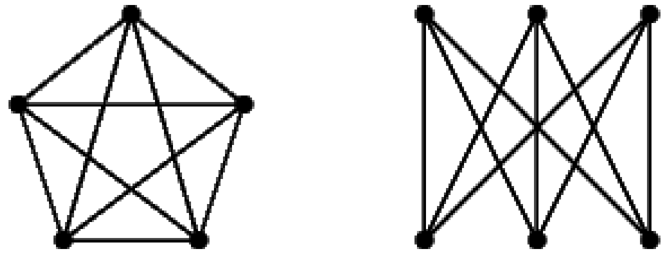


Fig – Five Planar graphs



K_5 $K_{3,3}$

Fig – Non Planar graphs

<u>Definition:</u>

A Jordan curve in the plane is a continuous non-self intersecting curve whose origin and terminus coincide.

C1 C2

Here C1 is not Jordan curve but C2 is a Jordan curve.

<u>Definition:</u>

If J is a Jordan curve in the plane then the part of the plane enclosed by J is called the interior of J and denoted by int J – we exclude from int J the points actually lying on J.
Similarly the part of the plane lying outside J is called the exterior of J and denoted by ext J.

Theorem 2.1.1 - Jordan Curve theorem

The Jordan curve theorem states that if J is a Jordan Curve and if x is a point in int J and y is a point in ext J then any line joining x to y must meet J at some point i.e, must cross J.

2.2-EULER'S FORMULA

Definition:

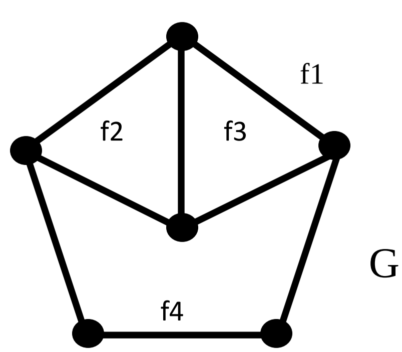
A plane graph G partitions the plane into a number of regions called the faces of G. More precisely, if x is a point on the plane which is not in G, i.e. is not a vertex of G or a point on any edge of G, then we define the face of G containing x to be the set of all points on the plane which can be reached from x by a line which does not cross any edge of G or go through any vertex of G.

Any plane graph has exactly one exterior face. The exterior face is unbounded. Any other face is bounded by a closed walk in the graph and is called an interior face.

The number of faces of a plane graph G is denoted by f(G) or just simply by f.

Example:

Consider the following graph G



G has 4 faces – f_1,f_2, f_3 and f_4 in which f_1 is the exterior face and f_2, f_3 and f_4 are interior faces. Hence f(G)=4.

Definition:

Let φ be the face of a plane graph G. We define the degree of φ , denoted by d(φ),to be the number of edges on the boundary of φ .
In the above example degree of f_4 is 5.

Note

d(φ) $\geq$3 for any interior face φ of a simple plane graph.

EULER'S FORMULA

Theorem 2.2.1

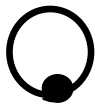
Let G be a connected plane graph, and let n,e and f denote the number of vertices, edges and faces of G, respectively. Then

$$n - e+f=2$$

Proof:
To prove this theorem, we use induction one, the number of edges of G.

If e=0 then G must have just one vertex, i.e. n=1 and one face, the exterior face, i.e. f=1.Thus n-e+f = 1-0+1=2 and so the result is true for e=0.

Now consider the case when e=1. Then the number of vertices of G is either 1 or 2,first possibly occurring when the edge is a loop. These two possibilities give rise to two faces and one face respectively, as shown in the figure



n=1

n=2

Thus,

n-e+f=1-1+2, in the loop case
=2-1+1, in the non-loop case
=2
as required.

Now suppose that the result is true for any connected plane graph G with e-1 edges (for a fixed e≥1).

Let us add one new edge k to G to form a connected super graph of G which we denote by G+k. There are three ways of doing this:

1) k is a loop, in which case we have created a new face (bounded by the loop) but the number of vertices remains unchanged, or

2) k joins two distinct vertices of G, in which case one of the faces of G is split into two, so again the number of faces has increased by one but the number of vertices has remained unchanged, or

3) k is incident with only one vertex of G in which case another vertex must be added, increasing the number of vertices by one, but leaving the number of faces unchanged.

Now let n', e' and f' denote the number of vertices, edges and faces in G and n, e and f denote the number of vertices, edges and faces in G+k. Then,

in case1

n-e+f = n'-(e'+1)+(f'+1)=n'-e'+f'

in case2

$n-e+f = n'-(e'+1)+(f'+1)= n'-e'+f$

in case3

$n-e+f =(n'+1)-(e'+1)+f'= n'-e'+f'$

and by our induction assumption, $n'-e'+f'=2$. Thus, in each case $n-e+f =2$.

Now any plane connected graph with e edges is of the form G+k, for some plane connected graph G with e-1 edges and a new edge k. Thus it follows by induction that the formula is true for all plane graphs.

Corollary 2.2.1

Let G plane graph with n vertices, e edges, f faces and k connected components. Then $n-e+f = k+1$.

Theorem 2.2.2

Let G be simple planar graph with n vertices and e edges, where $n \geq 3$.Then

$e \leq 3n-6$

Corollary 2.2.2

K_5 is non-planar.

Proof:

Here $n=5$ and $e =\frac{5*4}{2}=10$ so that $3n-6 =9$. Thus $e>3n-6$ and so, by the theorem, K_5 cannot be planar.

Corollary 2.2.3

$K_{3,3}$ is non-planar.

Proof:

Since $K_{3,3}$ is bipartite it contains no odd cycles and so in particular no cycle of length three. It follows that every face of plane drawing of $K_{3,3}$,if such exists must have at least have four boundary edges. Thus, using the argument of the proof of the above theorem, we get $b \geq 4f$ and then $4f \leq 2e$, i.e. $2f \leq e = 9$.
This gives $f \leq 9/2$.
However, by Euler's formula $f=2-n+e= 2-6+9 =5$,a contradiction.
Hence $K_{3,3}$ is non planar.

2.3-REPRESENTATION OF A PLANAR GRAPH

STRAIGHT-LINE REPRESENTATION

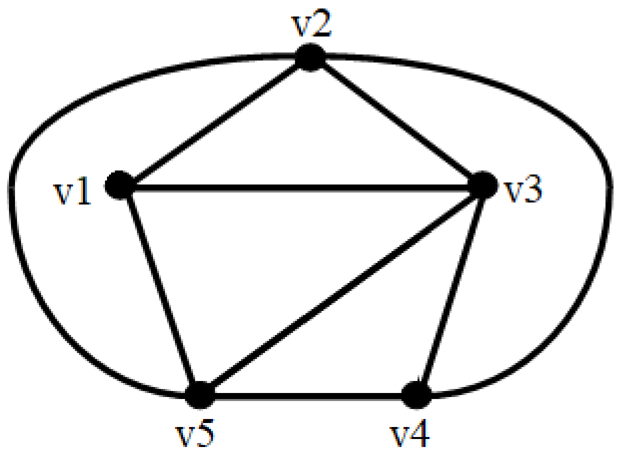
One's ability to draw a planar graph in a plane does not depend on his ability to draw many crooked lines through devious routes. The following is important and somewhat surprising result, due to Fary, tells us there is no need to bend edges in drawing a planar graph to avoid edge intersections.

Theorem 2.3.1

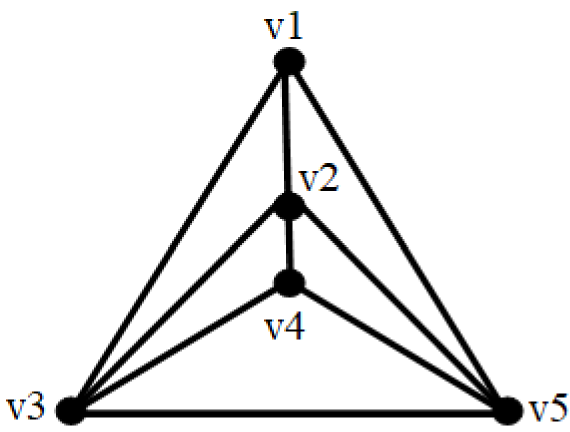
Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight-line segment.

Example:

Consider the graph



The straight-line representation of this graph is

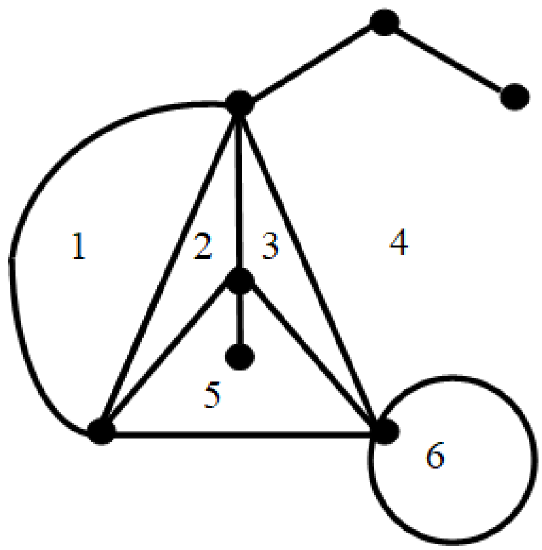


PLANE REPRESENTATION

A plane representation of a graph divides the plane into regions or faces. A region is characterized by the set of edges (or the set of vertices) forming its boundary.

Example:

The plane representation of the graph

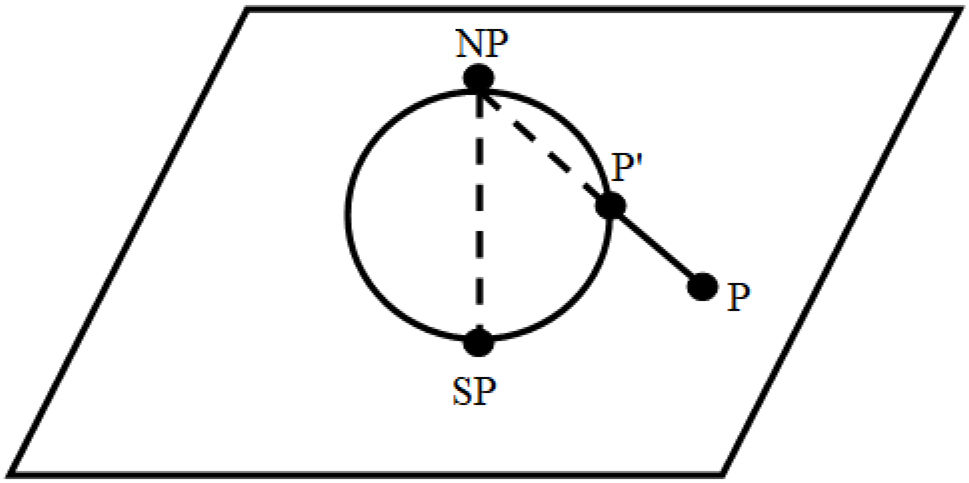


Note that a region is not defined in a non-planar graph or even in a planar graph not embedded in a plane. Thus, a region is a property of the specific plane representation of a graph.

We know that every planar graph has an exterior face or an exterior region. Like other regions, the infinite region also characterised by a set of edges. Clearly, by changing the embedding of a given paragraph, we can change the infinite region or the exterior region.

EMBEDDING ON A SPHERE

To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of a sphere. It is accomplished by stereographic projection of a sphere on a plane. Put the sphere on the plane and call the point of contact SP. At point SP, draw a straight line perpendicular to the plane and let the point where this line intersects the surface of the sphere be called NP.



Now, corresponding to any point P on the plane, there exists a unique point P' on the sphere and vice-versa, where P' is the point at which the straight line from P to point NP intersects the surface of the sphere. Thus, there is a one-to-one correspondence between the points of the sphere and finite points on the plane, and points at infinity in the plane correspond to the point NP on the sphere.

From this construction, it is cleared that any graph that can be embedded in a plane can also be embedded in the surface of the sphere and vice-versa. Hence the following theorem.

Theorem 2.3.2

A graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.

A planar graph embedded in the surface of a sphere divides the surface into different regions. Each region on the sphere is finite, the infinite region on the plane having been mapped onto the region containing the point NP. Now it is clear that by suitably rotating the sphere we can make any specified region map onto the infinite region on the plane. From this we obtain the following theorem.

Theorem 2.3.3

A planar graph may be embedded in a plane such that any specified region can be made the infinite region.

PLANE REPRESENTATION AND CONNECTIVITY

In a disconnected graph the embedding of each component can be considered independently. Therefore, it is clear that a disconnected graph is planar if and only if each of its components is planar. Similarly, in a separable (or t- connected) graph the embedding of each block can be considered independently. Hence a separable graph is planar if and only if each of its blocks is planar.

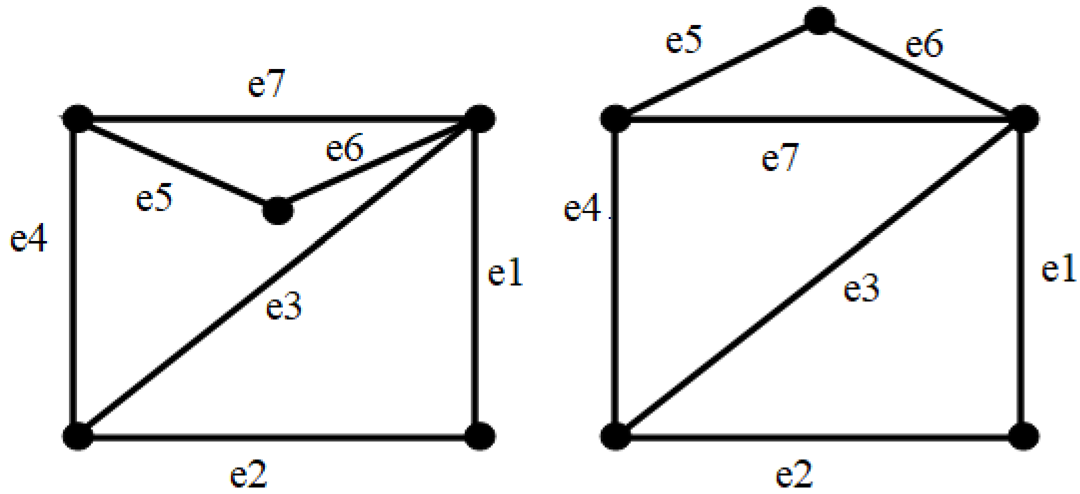
Therefore, in questions of embedding or planarity, one need to consider only non- separable graphs. Now we are going to consider the unique embedding of a non- separable planar graph on a sphere.

Definition:

Two embeddings of a planar graph on a sphere are not distinct if the embeddings can be made to coincide by suitably rotating one sphere with respect to other and possibly distorting regions (without letting a vertex cross an edge).If of all possible embeddings on a sphere no two are distinct, the graph is said to have a unique embedding on a sphere (or unique plane representation).

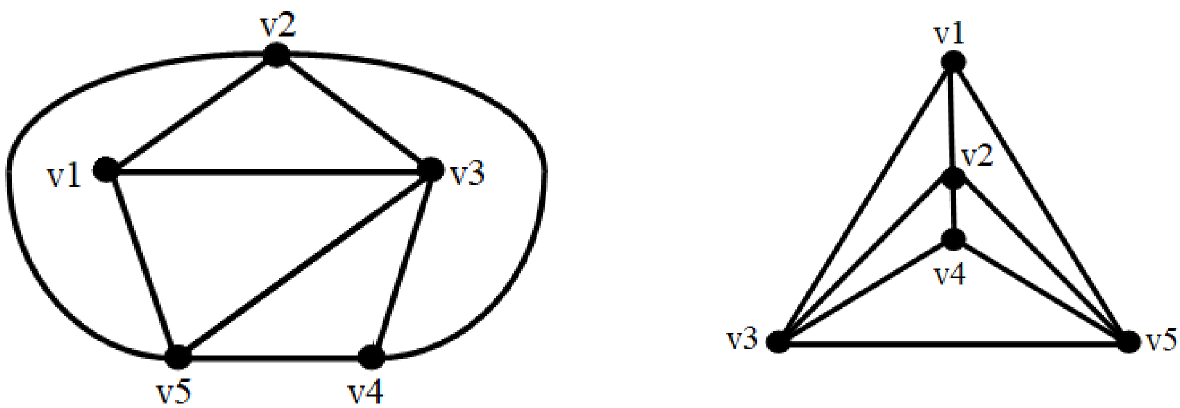
Example:

Consider two embeddings of the same graph given in the figure.



The embedding (b) has a region bounded with five edges, but embedding (a) has no regions with 5 edges. Thus, rotating two spheres on which (a) and (b) are embedded will not make them coincide. Hence two embeddings are distinct, and the graph has no unique plane representation.

Consider the two embeddings of another graph



These embeddings, when considered on a sphere, can be made to coincide.

The following theorem tells us exactly when a graph is uniquely embeddable in a sphere.

Theorem 2.3.4

The spherical embedding of every planar 3 – connected graph is unique.

2.4 – TYPES OF PLANAR GRAPH

MAXIMAL PLANAR GRAPHS

A simple graph is called maximal planar if it is planar but adding any edge (on the given vertex set) would destroy that property. Every face of a maximal planar graph is bounded by three edges. Hence it is called plane triangulation or triangulation of the sphere. Every maximal planar is 3- vertex connected.

Example:

The Goldner- Harary graph is maximal planar, which contains 11 vertices ,27 edges with radius and diameter 2

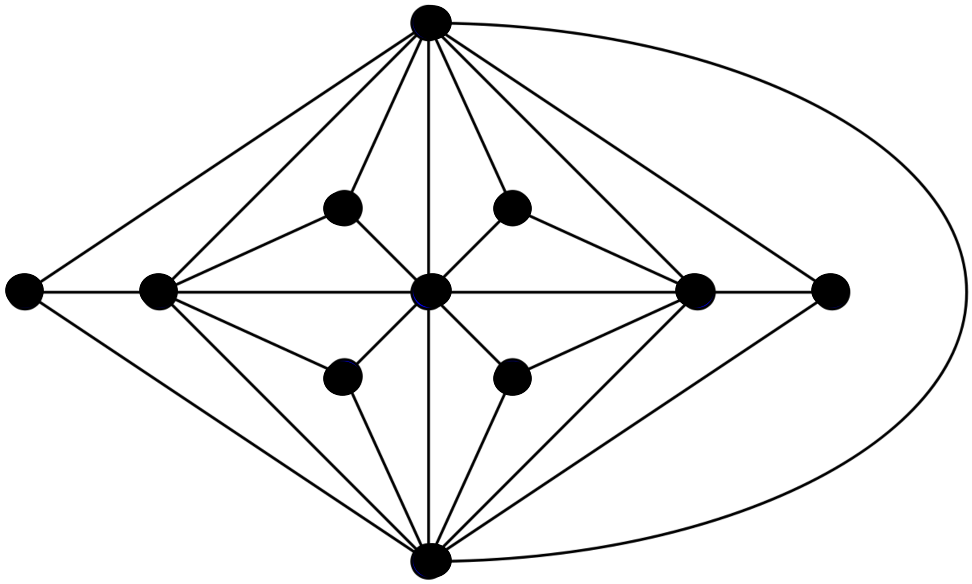


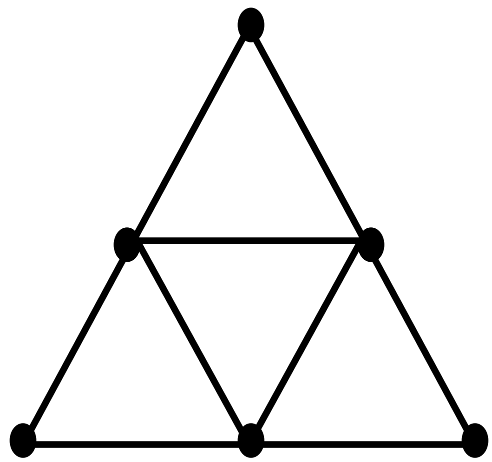
Fig:- Goldner- Harary graph

OUTER PLANAR GRAPHS

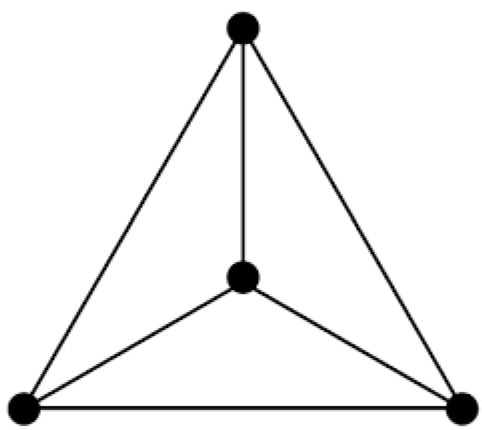
Outer planar graphs are graphs with an embedding in the plane such that all vertices belong to the unbounded face of embedding. Every outer planar graph is planar, but the converse is not true.

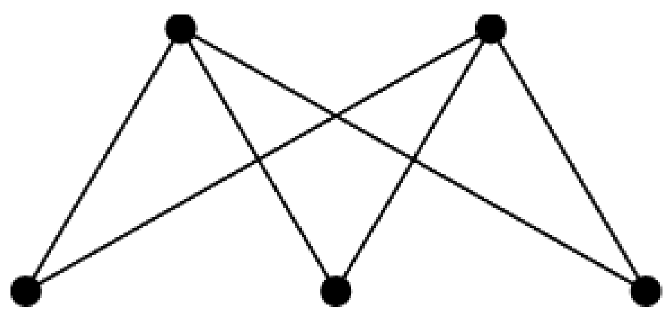
Example:

The following graph is an outer planar graph.



K_4 and $K_{2,3}$ are planar but not outer planar.





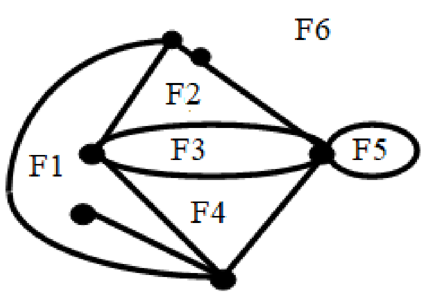
CHAPTER -3

DUAL GRAPHS

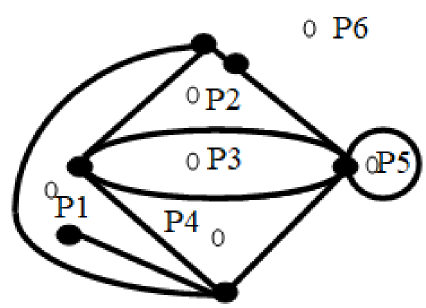
3.1-DUAL OF A PLANE GRAPH

Definition:

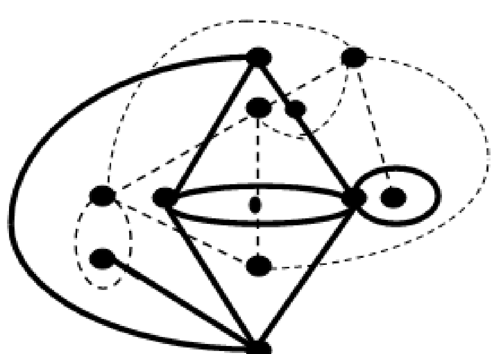
Consider the plane representation of a graph in figure (a) with six regions or faces F_1 F_2 F_3 F_4 F_5 F_6. Let us place six points P_1 P_2 P_3 P_4 P_5 P_6 on figure (b). Next let us join these 6 points according to the following procedure.



(a)



(b)



(c)

If 2 regions F_i and F_j are adjacent (have a common edge) draw a line joining points P_i and P_j that intersects the common edge between F_i and F_j exactly once. If there is more than one edge common between F_i and F_j draw one line between points P_i and P_j for each of the common edges. For an edge e lying entirely in one region say F_k, draw self loop at point P_k intersecting e exactly once. By the procedure, we obtain a new graph G* consisting of 6 vertices and of edges joining these vertices. Such a graph G* is called DUAL [or strictly speaking a geometric dual] of G.

Clearly, there is a one-to-one correspondence between the edges of graph G and its dual G* - one edge of G* intersecting one edge of G

Following are some simple observation that can be made about the relationship between a planar graph G and it's dual G*

1. A pendant edge in G yields a self loop in G*

2. An edge forming a self loop in G yields a pendant edge in G*

3. Parallel edges in G produces edges in series in G*

4. Edges that are in series in G produce parallel edges in G*

5. The number of edges in constituting the boundary of a region Fi in G is equal to the degree of the corresponding vertex Pi in G*

6. Graphs G* is also embedded in the plane and is therefore planar.

7. If n, e, f, r and u denote as usual the number of vertices, edges, regions, rank and nullity of a connected planar graph G, and if n*,e*,f*,r* and u* are the corresponding numbers in dual graphs G*, then,

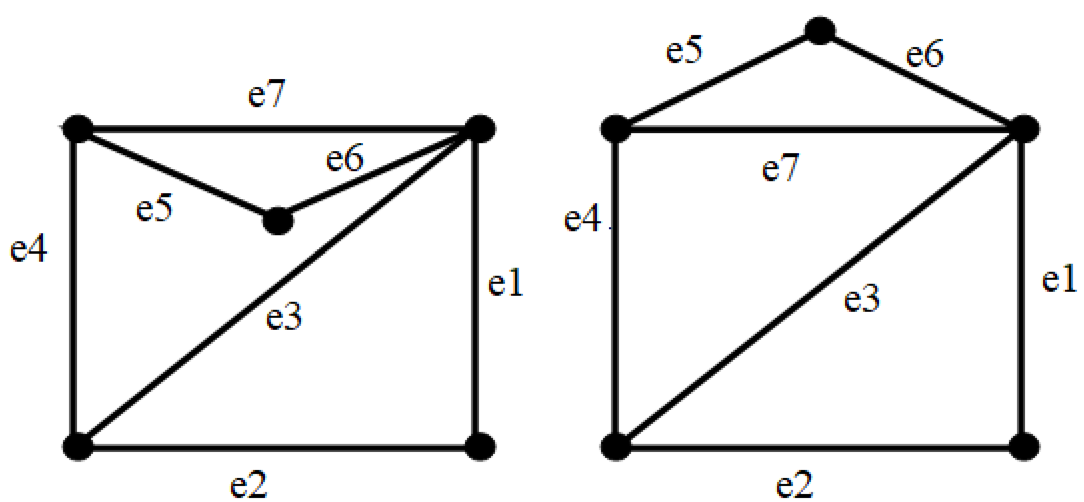
$n^* = f$
$e^* = e$
$f^* = n$

Using the above relationship, one can immediately get,
$r^* = u$
$u^* = r.$

3.2-UNIQUENESS OF DUAL GRAPHS

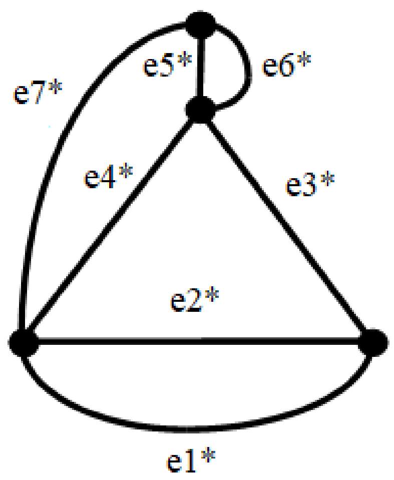
Because the dual graph depends on a particular embedding, the dual graph of a planar graph is not unique, in the sense that the same planar graph can have non- isomorphic dual graphs.

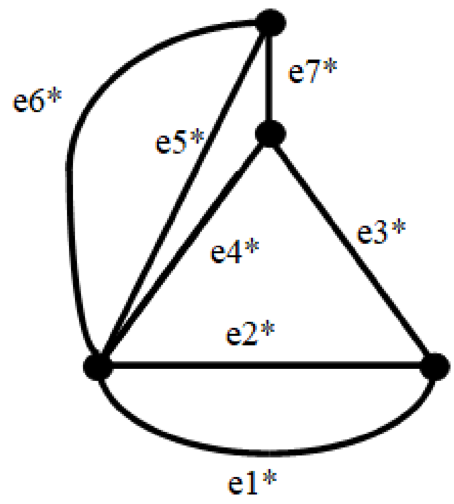
Consider the two isomorphic graphs G_1 and G_2.



G_1 G_2

Actually, this is two distinct plane representations of the same graph. The duals of these graphs are





G_1* G_2*

Clearly, these two duals are non-isomorphic as G_2^* has a vertex of degree 5 which is not present in G_1^*.

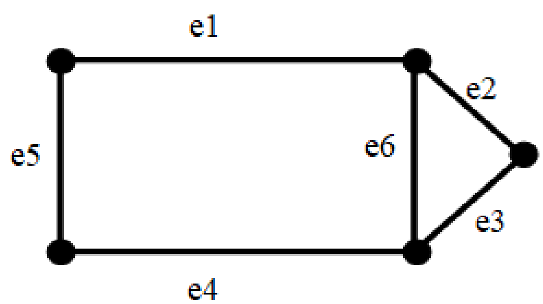
A planar graph G has a unique dual if and only if graph G has a unique plane representation or graph G can be uniquely embedded on a sphere. As a 3-connected planar graph has a unique embedding on a sphere, so its dual must be unique. On the other hand, all duals of a 3-connected graph are isomorphic.

3.3-DUAL OF A SUBGRAPH

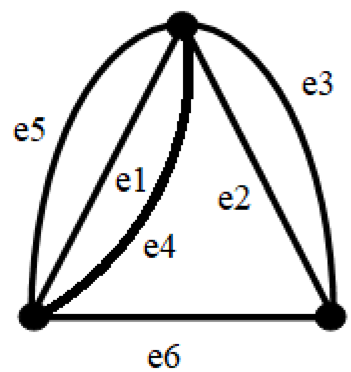
Let G be a planar graph and G^* be its dual. Let e be an edge in G and the corresponding edge in G^* be e^*. Suppose that we delete edge e from G and then try to find the dual of G-e. If edge e was on the boundary of two regions, removal of e would merge these two regions into one. Thus the dual $(G\text{-}e)^*$ can be obtained from G^* by deleting the corresponding edge e^* and then fusing the two end vertices of e^* in $G^*\text{-}e^*$. On the other hand, if edge e is not on the boundary, e^* forms a self-loop. In that case $G^*\text{-}e^*$ is same as $(G\text{-}e)^*$. Thus if a graph G has a dual G^*, the dual of any subgraph of G can be obtained by successive application of this procedure.

Example:

Consider the graph G and its dual G^*

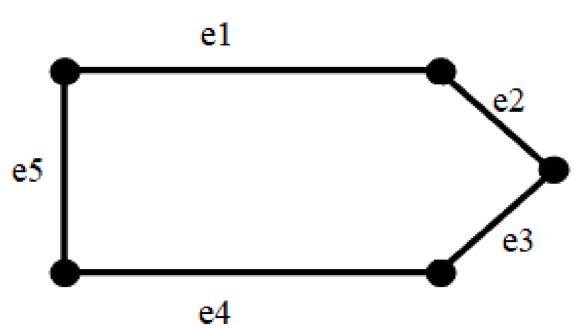


G

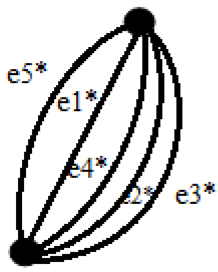


G*

If we delete the edge e_6 from G, the graph will be



The dual of this graph is obtained by deleting the edge e_6^* from G* and then fusing the two end vertices of e_6^* in G^*-{e_6^*}.



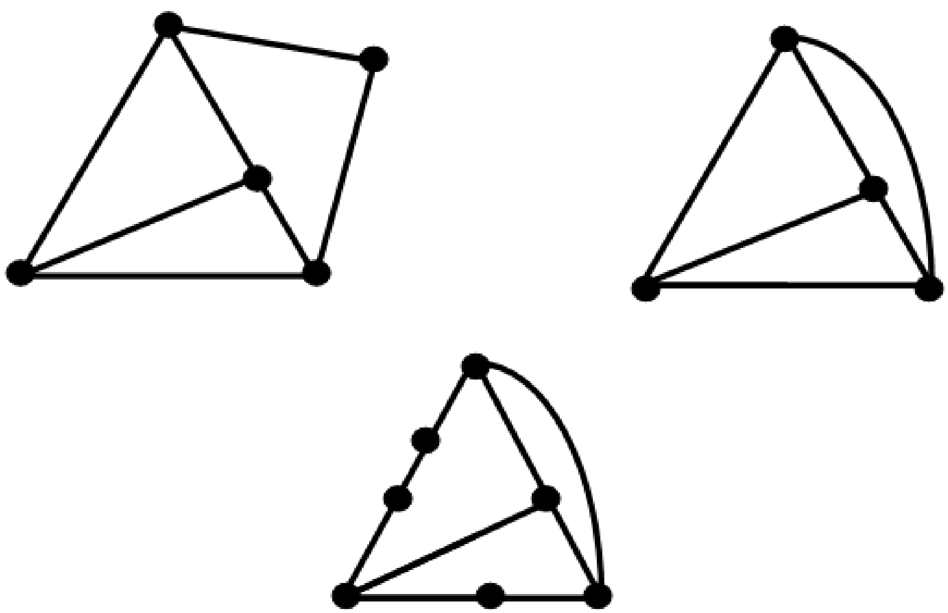
3.4-DUAL OF A HOMEOMORPHIC GRAPH

Definition:

Two graphs are said to be homeomorphic if one graph can be obtained from the other by creation of edges in series (i.e. by insertion vertices of degree two) or by the merging of edges in series.

Example:

The following three graphs are homeomorphic to each other.



DUAL OF A HOMEOMORPHIC GRAPH

Let G be a planar graph and G^* be its dual. Let e be an edge in G, and the corresponding edge in G^* be e^*. Suppose that we create an additional vertex in G by introducing a vertex of degree two in edge e (i.e. e now becomes two edges in series). This will simply add an edge parallel to e* in G*. Likewise, the reverse process of merging two edges in series will simply eliminate one of the corresponding parallel edges in G*.Thus if a graph G has a dual G* ,the dual of any graph homeomorphic to G can be obtained from G* by the above procedure

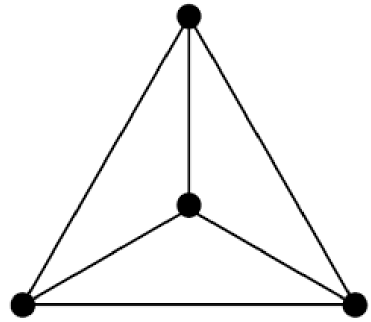
3.5- SELF DUAL GRAPHS

Definition:

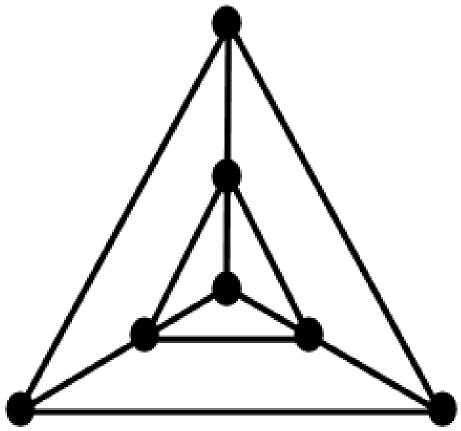
If a planar graph G is isomorphic to it's own dual, it is called a self dual graph.

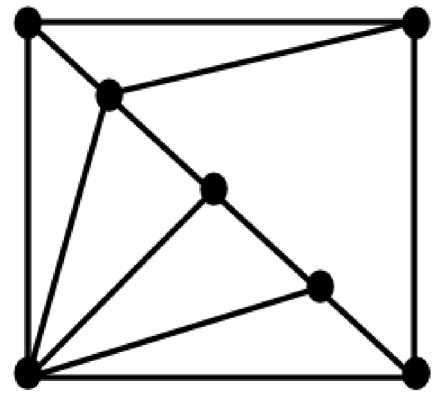
Example:

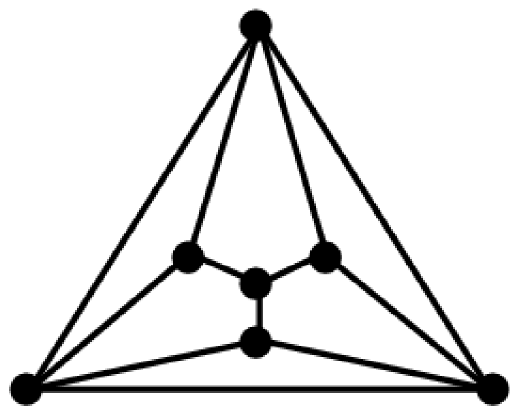
The complete graph on four vertices K_4 is a self-dual graph.



Wheel graphs are self dual. Naturally the skeleton of a self dual polyhedron is a self dual graph. Since the skeleton of a pyramid is a wheel graph. It follows that pyramids are also self dual.







Theorem 3.5.1 - Kuratowski's Theorem

A necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski's two graphs or any graph homomorphic to K_5 or $K_{3,3}$.

Theorem 3.5.2

A graph has a dual if and only if it is planar.

Proof:

We have only to prove that a non-planar graph does not have a dual. Let G be a non-planar graph. Then according to Kuratowski's theorem, G contains K_5 or $K_{3,3}$, or a graph isomorphic to either of these

We know that a graph G has a dual only if every subgraph H of G and every subdivision of G has a dual. Thus, to prove the result, we show that neither K_5 nor $K_{3,3}$ has a dual.

Suppose that $K_{3,3}$ has a dual D. Then the bonds in $K_{3,3}$ correspond to cycles in D and vice versa. Since $K_{3,3}$ has no bond consisting of two edges, D has no cycle consisting of two edges. That is, D contains no pair of parallel edges. Since every cycle in $K_{3,3}$ is of length 4 or 6, D has no bond with less than 4 edges. Therefore, the degree of every vertex in D is at least 4. As D has no parallel edges and degree of every vertex is at least 4, D has at least 5 vertices each of degree 4 or more. That is, D has at least (5*4)/2=10 edges. This is a contradiction to the fact that $K_{3,3}$ has 9 edges. Hence $K_{3,3}$ has no dual.

Suppose K_5 has a dual H. We note that K_5 has:

1) 10 edges
2) No pair of parallel edges
3) No bond with 2 edges
4) Bonds with only 4 or 6 edges

Therefore, graph H has

1) 10 edges
2) No vertex with degree less than 3
3) No bond with 2 edges
4) Cycles of length 4 or 6 only

Now, graph H contains a cycle of length 6 and no more than 3 edges can be added to a hexagon without creating a cycle of length 3, or a pair of parallel edges. Since both of these are not present in H, and H has 10 edges, there must be at least 7 vertices in H. The degree of each of the vertices is at least 3. This implies that H has at least 11 edges, which is a contradiction. Hence K_5 has no dual.

CHAPTER -4

APPLICATION

Planarity and the other related concepts are useful in many practical situations. For example, in the design of a printed circuit board and the three utilities problem, planarity is used.

The study of two-dimensional images often results in problems related to planar graphs, as does the solution of many problems on the two-dimensional surface of our Earth. Many three-dimensional graphs arise in scientific and engineering problems. These often come from well-shaped meshes, which share many properties with planar graphs. While planar graphs were introduced for practical reasons, they possess many remarkable mathematical properties.

CONCLUSION

Planar Graphs constitute quite simple class of graphs, much simpler than the class of all graphs. So,as the science frequently does, if some algorithmic problem cannot be solved efficiently for all interesting inputs, we can at least strive to solve it for some of the inputs. Indeed, many problems that are hard for general graphs turn to possess polynomial-time algorithms when they are restricted to planar graphs due to their sparsity.

Although being almost too structurally simple, planar graphs, should not be considered non-applicable to real life. For example, the task of large electronic circuit layout employs planar graph layout. Many algorithms for graph drawing, although targeting non-planar graphs, have a planar oriented core, i.e. try to make an input graph planar, then draw it, and then get back to the original graph.

Planarity is one of the central notions of the whole graph theory, so just purely from the theoretical point of view it is interesting to consider planar graphs in algorithmic frame work. While there is a bunch of existing algorithms for testing planarity, the topic is still being researched and new optimizations are being discovered.

REFERENCES

1) John Clark, Derek Allen Holton, **A First Look At Graph Theory**, World Scientific Publishing Co. Pvt. Ltd., 1991

2) Narsingh Deo, **Graph Theory With Applications To Engineering And Computer Science**, Prentice Hall of India Pvt. Ltd., 1974