JORDAN CANONICAL FORM

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CERTIFICATE

This is to certify that the project report titled "JORDAN CANONICAL FORM" submitted by RIZLA.M.A(Reg no. 170021032424), ADHILA LATHEF.K.A(Reg no. 170021032392) and ELIZABETH MARIYA TEENA THOMAS(Reg. no:170021032408) towards partial fulfilment of the requirements for the award of Degree of Bachelor of Science in Mathematics is a bonafide work carried out by them during the academic year 2017-2020.

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DECLARATION

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JORDAN CANONICAL FORM

CONTENTS

INTRODUCTION

Jordan canonical form is a representation of a linear transformation over a finite dimensional complex vector space by a particular kind of upper triangular matrix. Every such linear transformation has a unique Jordan canonical form, which has useful properties Jordan canonical form is a representation of linear transformation over a finite: it is easy to describe and well suited for computations.

Less abstractly one can speak of the Jordan canonical form of a square matrix. Every square matrix is similar to a unique matrix in Jordan canonical form. Since similar matrices correspond to representation of the same linear transformation with respect to different basis by the change of basis theorem.

Jordan canonical form can be thought as a generalization of diagonalizability to arbitrary linear transformation [or matrices] indeed the Jordan canonical form of a diagonalizable linear transformation [or a diagonalizable matrix] is a diagonal matrix.

Chapter: 1
PRELIMINARIES

SECTION: 1.1

EIGENVECTORS AND EIGENVALUES

Let **A** be a $n \times n$ matrix λ a scalar be an eigenvalue of **A** we shall mean a scalar λ for which there exist a non-zero $n \times 1$ matrix such that $\mathbf{A}x = \lambda x$, such a column matrix x is called eigenvector associated with λ.

SECTION: 1.2

SIMILAR MATRICES

Definition: 1

A matrix **A** is similar to a matrix **B** if there exists an invertible matrix **P** such that,

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP} \qquad (1)$$

If we pre-multiply (1) by **P**, it follows that **A** is similar to **B** if and only if there exist a non-singular matrix **P** such that,

$$\mathbf{PA} = \mathbf{BP} \qquad (2)$$

Furthermore if we post-multiply (2) by $\mathbf{P}^{-1}$ .we see that **A** is similar to **B** if and only if **B** is similar **A**.

Example: 1

Determine whether $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$ is similar to $\mathbf{B} = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$

Solution:

A will be similar to **B** if and only if there exists a non-singular matrix **P** such that (2) is satisfied. Designate **P** by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then **PA=BP** implies that

$$\begin{bmatrix} (4a-2b) & (3a-b) \\ (4c-2d) & (3c-d) \end{bmatrix} = \begin{bmatrix} (5a-4a) & (5b-4d) \\ (3a-2c) & (3b-2d) \end{bmatrix}$$

Equating corresponding elements, we find that the elements of **P** must satisfy the four equations:

-a-2b=4c=0,

3a-6b+4d=0,

-3a+6c-2d=0,

-3b+3c+d=0.

A solution to this set of equations is a=-2d/3, b=1/d, with c=0, d arbitrary.

Thus **P** = $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{d}{3}\begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$

P is invertible if d ≠ 0. Thus by choosing d ≠ 0, we obtain an invertible matrix **P** that satisfies (2) which implies that **A** is similar to **B**.

SECTION: 1.3

DIAGONALIZABLE MATRICES

Definition: 2

A matrix is diagonalizable if it is similar to a diagonalizable matrix.

Example: 2

Determine whether **A** = $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ is diagonalizable.

Solution:

The eigenvalues of **A** are -1 and 5. Since the eigenvalues are distinct. Their respective eigenvector $x_{-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are linearly independent, hence the matrix is diagonalizable.

We choose their $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ or $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

Making the first choice, we find that

$\mathbf{D} = \mathbf{M^{-1}AM}$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

Making the choice we find

$\mathbf{D} = \mathbf{M^{-1}AM}$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}.$$

Chapter: 2

GENERALIZED EIGENVECTORS

SECTION: 2.1

GENERALIZED EIGENVECTORS

In the previous chapter, we showed that if a matrix **A** has linearly independent eigenvector associated with it and hence is diagonalizable, then well-defined matrix functions of **A** can be computed. We now generalize our analysis and obtain similar results begin by generalizing the concept of the eigenvector.

Definition: 1

A vector $\mathbf{x}_m$ is a *generalized eigenvector of type m* corresponding to the matrix **A** and the eigenvalue λ if $(\mathbf{A} - \lambda\mathbf{I})^m\mathbf{x}_m = 0$ but $(\mathbf{A}-\lambda\mathbf{I})^{m-1} \neq 0$.

For example

If $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, then $\mathbf{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a generalized eigenvector of type 3 corresponding to λ = 2, since

$$(\mathbf{A}\text{-}2\mathbf{I})^3\mathbf{x}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But,

$$(\mathbf{A}\text{-}2\mathbf{I})^2\mathbf{x}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq 0$$

Also,

$\mathbf{x_2} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is a generalized eigenvector of type 2 corresponding to λ = 2

$$(\mathbf{A}\text{-}2\mathbf{I})^2\mathbf{x}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But,

$$(\mathbf{A}\text{-}2\mathbf{I})^1\mathbf{x}_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq 0$$

Furthermore,

$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a generalized eigenvector of type 1 corresponding to the eigenvalue λ = 2 since $(\mathbf{A}\text{-}2\mathbf{I})^1\mathbf{x}_1 = 0$ but $(\mathbf{A}\text{-}2\mathbf{I})^0\mathbf{x}_1 = \mathbf{x}_1 \neq 0$.

We note for a reference that a generalized eigenvector of type 1 is in fact an eigenvector.

Example: 2.1.1

It is known that the matrix $\mathbf{A} = \begin{bmatrix} 5 & 0 & 2 \\ 2 & 1 & 1 \\ -5 & 1 & -1 \end{bmatrix}$ has a generalized eigenvector of type 2 corresponding to λ = 2.Find it?

Solution:

We seek a vector $\mathbf{x_2}$ such that $(\mathbf{A}\text{-}2\mathbf{I})^2\mathbf{x}_2 = 0$ and designate $\mathbf{x_2}$ by $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then

$$(\mathbf{A}\text{-}2\mathbf{I})^2\mathbf{x}_2 = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + 2y \\ -x + 2y \\ 2x - 4y \end{bmatrix} \text{ and}$$

$$(\mathbf{A}\text{-}2\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 3 & 0 & 2 \\ 2 & -1 & 1 \\ -5 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x+2z \\ 2x-y+z \\ -5x+y-3z \end{bmatrix}$$

For $(\mathbf{A}\text{-}2\mathbf{I})^2\mathbf{x}_2 = 0$, it follows that $x = 2y$.Using this result we obtain

$$(\mathbf{A}\text{-}2\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 6y+2z \\ 3y+z \\ -9y-z \end{bmatrix}$$

Since this vector must not be zero, it follows that $z \neq -3y$.There are infinitely many values of x, y, z that simultaneously satisfy the requirements $x = 2y$ and

$z \neq -3y$ (for instance $x = 2$, $y = 1$, $z = 4$).The simplest choice is $x = y = 0$,

$z = 1$.

Thus, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a generalized eigenvector of type 2 corresponding $\lambda = 2$.

Example: 2.1.2

It is known the matrix $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ has a generalized eigenvector of type 2, corresponding to $\lambda = 4$. Find it?

Solution:

We seek a vector $\mathbf{x}_2$ such that $(\mathbf{A}\text{-}4\mathbf{I})^2\,\mathbf{x}_2 = 0$ and $(\mathbf{A}\text{-}4\mathbf{I})\mathbf{x}_2 \neq 0$

Designate $\mathbf{x}_2$ by $\begin{bmatrix} x \\ y \end{bmatrix}$, then

$$(\mathbf{A}\text{-}4\mathbf{I})^2\mathbf{x}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus we see that every vector has the property that $(\mathbf{A}-4\mathbf{I})^2\mathbf{x}_2 = 0$. Hence need place no restrictions on either x or y to achieve this result

$$(\mathbf{A}-4\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

Cannot be the zero vector, it must be the case that $y \neq 0$.Thus by choosing $x = 0$ and $y = 1$, we obtain

$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a generalized eigenvector of type 2 corresponding to λ=4

SECTION: 2.2

CHAINS

Definition: 2

Let x_m be a generalized eigenvector of type m corresponding to the matrix **A** and the eigenvalue λ.

The *chain generated by* $\mathbf{x}_m$ is a set of vectors $\{\mathbf{x}_m \ \mathbf{x}_{m-1} \ . \ . \ . \ \mathbf{x}_1\}$

given by

$\mathbf{x}_{m-1} = (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}_m$

$\mathbf{x}_{m-2} = (\mathbf{A}-\lambda\mathbf{I})^2\mathbf{x}_m = (\mathbf{A}-\lambda\mathrm{I})\mathbf{x}_{m-1}$

$\mathbf{x}_{m-3} = (\mathbf{A}-\lambda\mathbf{I})^3\mathbf{x}_m = (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}_{m-2}$

.

.

.

$\mathbf{x}_1 = (\mathbf{A}-\lambda\mathbf{I})^{m-1}\mathbf{x}_m = (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}_2$

Thus in general,

$$\mathbf{x}_j = (\mathbf{A}-\lambda\mathbf{I})^{m-j}\mathbf{x}_m = (\mathbf{A}-\lambda\mathbf{I})^{m}\mathbf{x}_{j+1} \ ;\ j = 1,2,\ldots,m-1 \qquad (1)$$

Theorem: 2.2.1

$\mathbf{x}_j$ *(given by (1)) is a generalized eigenvector of type j corresponding to the eigenvalue* λ

Proof:

Since $\mathbf{x}_m$ is a generalized eigenvector of type *m*, $(\mathbf{A}-\lambda\mathbf{I})^m\mathbf{x}_m = 0$ and

$(\mathbf{A}-\lambda\mathbf{I})^{m-1}\mathbf{x}_m \neq 0$.

Thus using (1) we find that,

$(\mathbf{A}-\lambda\mathbf{I})^j\mathbf{x}_j = (\mathbf{A}-\lambda\mathbf{I})^j(\mathbf{A}-\lambda\mathbf{I})^{m-j}x_m = (\mathbf{A}-\boldsymbol{\lambda}\mathrm{I})^m\mathbf{x}_m = 0$ and

$(\mathbf{A}-\lambda\mathbf{I})^{j-1}\mathbf{x}_j = (\mathbf{A}-\lambda\mathbf{I})^{j-1}(\mathbf{A}-\lambda\mathbf{I})^{m-j}\mathbf{x}_m = (\mathbf{A}-\lambda\mathbf{I})^{m-1}\mathbf{x}_m \neq 0$

Which together imply *theorem: 1*

Thus once we have found a generalized eigenvector of type *m*, it is simple to obtain a generalized eigenvector of any type less than *m*.

For example, we found in the previous section that

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

is a generalized eigenvector of type 3, for

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \text{ corresponding to } \lambda = 5.$$

Using *theorem: 1*, we now can state that

$$\mathbf{x}_2 = (\mathbf{A}-5\mathbf{I})\mathbf{x}_3 = \begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$ is a generalized eigenvector of

type 2 corresponding to $\lambda = 5$ while,

$\mathbf{x}_1 = (\mathbf{A}\text{-}5\mathbf{I})^2\mathbf{x}_3 = \begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a generalized eigenvector

of type 1, hence an eigenvector corresponding to $\lambda = 5$.

The set $\{\mathbf{x}_3, x_2, \mathbf{x}_1\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is the chain generated by $\mathbf{x}_3$.

The value of the chain is hinted at by the following theorem.

Theorem: 2.2.2

A chain is a linearly independent set of vectors.

Proof:

Let $\{\mathbf{x}_m, \mathbf{x}_{m-1}, \ldots, \mathbf{x}_1\}$ be a chain generated from $\mathbf{x}_m$ a generalized eigenvector of type m corresponding to the eigenvalue λ of a matrix $\mathbf{A}$ and consider the vector equation

$$c_m\mathbf{x}_m + c_{m-1}\mathbf{x}_{m-1} + \ldots + c_1\mathbf{x}_1 \qquad (*)$$

In order to prove that this chain is linearly independent set, we must show that the only constants satisfying the above equation are $c_m = c_{m-1} = \ldots = c_1 = 0$.

Multiply the equation by $(\mathbf{A}-\lambda\mathbf{I})^{m-1}$ and note that for $j = 1,2, \ldots, m - 1$.

$(\mathbf{A}-\lambda\mathbf{I})^{m-1} c_j \mathbf{x}_j = c_j(\mathbf{A}-\lambda\mathbf{I})^{m-j-1}(\mathbf{A}-\lambda\mathbf{I})^j \mathbf{x}_j$

$= c_j(\mathbf{A}-\lambda\mathbf{I})^{m-j-1} \times 0$ (Since $\mathbf{x}_j$ is generalized eigenvector of type j)

$= 0$

Thus,

$c_m(\mathbf{A}-\lambda\mathbf{I})^{m-1}\mathbf{x}_m = 0$

However, since $\mathbf{x}_m$ is a generalized eigenvector of type m, $(\mathbf{A}-\lambda\mathbf{I})^{m-1}\mathbf{x}_m \neq 0$, from which it follows that $c_m = 0$.Substituting $c_m = 0$ in (*) and then multiply (*) by $(\mathbf{A}-\lambda \mathrm{I})^{m-2}$, we find by similar reasoning that $c_{m-1} = 0$.Continuing this process, we finally obtain $c_m = c_{m-1} = \ldots = c_1 = 0$, which implies that the chain is linearly independent.

Chapter: 3

JORDAN CANONICAL FORM

SECTION: 3.1

CANONICAL BASIS

Theorem: 1

*Every $n \times n$ matrix **A** possesses n linearly independent generalized eigenvectors, henceforth abbreviated liges. Generalized eigenvectors corresponding to distinct eigenvectors are linearly independent. If λ is an Eigen value of **A** of multiplicity v, then **A** will have v, liges corresponding to λ.*

For every given matrix **A**, there are infinitely many ways to pick the n liges. If they are chosen in a particularly judicious manner, we can use these vectors to show that **A** is similar to an "almost diagonal matrix".

Definition:

A set of n liges (Linearly independent generalized eigenvectors) is a canonical basis for an $n \times n$ matrix if the set is composed entirely of chains.

Thus, once we have determined that a generalized eigenvector of type m is in a canonical basis. It follows that m-1 vectors $\mathbf{x}_{m-1}, \mathbf{x}_{m-2}, \ldots, \mathbf{x}_1$ that are in the chain generated by $\mathbf{x}_m$.

Let λ_i be an eigenvalue of **A** of multiplicity v. First find the rank of the matrixes

$(\mathbf{A}-\lambda_i \boldsymbol{I})$, $(\mathbf{A} - \lambda_i \boldsymbol{I})^2$, ...,$(\mathbf{A}- \lambda_i \boldsymbol{I})^m$. The integer m is determined to be first integer for which $(\mathbf{A} - \lambda_i \boldsymbol{I})^m$ has rank $n - v$ (n being the no of rows and columns of **A** ie, **A** is $n \times n$).

Example: 1

Determine m corresponding to $\lambda_i = 2$ for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Solution:

n = 6 and the eigenvalue $\lambda_i = 2$ has multiplicity v=5.Hence $n - v = 1$.

$$(\mathbf{A}\text{-}2\mathbf{I}) = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

has rank 4.

$$(\mathbf{A}\text{-}2\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

has rank 2.

$$(\mathbf{A}\text{-}2\mathbf{I})^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

Has rank 1= $n - v$.

Therefore, corresponding to $\lambda_i = 2$, we have m=3

Now, define

$\rho_k = r((\mathbf{A}-\lambda_1\mathbf{I})^{k-1}) - r((\mathbf{A}-\lambda_1\mathbf{I})^k)$ $\qquad k = 1,2,3,\ldots,m.$

ρ_k designates the no of liges of type k corresponding to the eigenvalue λ_i that will appear in a canonical basis of A.

$$r(\mathbf{A}-\lambda_i\mathbf{I})^0 = r(\mathbf{I}) = n.$$

Example: 2

Determine how many eigenvectors of type of each type corresponding to $\lambda_1 = 2$ will appear in a canonical basis for the example1.

Solution:

Using the results of example 1, we have that

$$\rho_3 = r(\mathbf{A} - 2\mathbf{I})^2 - r(\mathbf{A} - 2\mathbf{I})^3 = 2 - 1 = 1$$

$$\rho_2 = r(\mathbf{A} - 2\mathbf{I})^1 - r(\mathbf{A} - 2\boldsymbol{I})^2 = 4 - 2 = 2$$

$$\rho_1 = r(\mathbf{A} - 2\mathbf{I})^0 - r(\mathbf{A} - 2\mathbf{I})^1 = 6 - 4 = 2.$$

Thus a canonical basis for the matrix given in example 1 will have, corresponding to λ_1=2 , one generalized eigenvector of type 3, two liges of types 2 and two liges of type 1.

Example: 3

Find a canonical basis for the **A** given in example 1.

Solution:

We first find liges corresponding to $\lambda_i = 2$

From example 2, we know that there is one generalized eigenvector of

type 3;

we find the vector to be,

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then we can obtain $\mathbf{x_2}$ and $\mathbf{x_1}$ as generalized eigenvectors of type 2 and 1 , where

$$\mathbf{x_2} = (\mathbf{A}\text{-}2\mathbf{I})\mathbf{x_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathbf{x_1} = (\mathbf{A}\text{-}2\mathbf{I})\mathbf{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From example 2 we know that a canonical basis for A also has two liges of type 2 corresponding to λ_i =2. We already found one of these vectors to be $\mathbf{x_2}$.

Therefore we seek a generalized eigenvector $y_2$of type 2that is linearly independent of $\{\mathbf{x_3}, \mathbf{x_2}, \mathbf{x_1}\}$.

$$\text{Designate } \mathbf{y_2} = \begin{bmatrix} u_2 \\ v_2 \\ w_2 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

We find that in order for y_2 to be a generalized eigenvector of type 2,

$w_2 = x_2 = 0, or\ v_2 or\ y_2$ must be non zero ,and $v_2$and x_2 are arbitrary.

If we pick $u_2 = w_2 = x_2 = y_2 = z_2 = 0, v_2 = 1$,

$$\mathbf{y_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

As a generalized eigenvector of type 2. This vector, however, is not linearly independent of $\{\mathbf{x_3}, \mathbf{x_2}, \mathbf{x_1}\}$ since $\mathbf{y_2} = \boldsymbol{x_2} + \boldsymbol{x_1}$.

If instead we choose $u_{2=} v_2 = w_2 = x_2 = z_2$=0, $y_2 = 1$,

we obtain $\mathbf{y_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

which satisfies all the necessary requirement.

(Note that there are many other adequate choices for $y_{2.}$ in particular we could have chosen $u_2 = w_2 = x_2 = z_{2,} v_2 = y_2 = 1$)

$$\boldsymbol{y_1} = (\mathbf{A} - 2\mathbf{I})\ \boldsymbol{y_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

is a generalized eigenvector of type 1.

From example 2, we know that

Having found all the liges corresponding to λ_1 = 2, we direct our attention to the liges corresponding to λ_2 = 4. From our previous discussion, we know that the only generalized eigenvectors corresponding to $\lambda_2 = 4$ is the eigenvector itself,

which we determine to be

$\mathbf{z_1}=\begin{bmatrix}0\\0\\0\\1\\2\\4\end{bmatrix}$

Thus a canonical basis for **A** is $\{\mathbf{x_3}, \mathbf{x_2}, \mathbf{x_1}, \mathbf{y_2}, \mathbf{y_1}, \mathbf{z_1}\}$ note that due to theorem 1, we do not have to check whetherz_1 is linearly independent of $\{\mathbf{x_3}, \mathbf{x_2}, \mathbf{x_1}, \mathbf{y_2}, \mathbf{y_1}\}$.

Since $\mathbf{z_1}$ corresponds to λ_2 and all the other vectors correspond to λ_1 where $\lambda_1 \neq \lambda_2$, linear independence is guaranteed.

Example:4

Find a canonical basis for

$$\mathbf{A} = \begin{bmatrix}1 & 1 & 0 & -1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{bmatrix}$$

Solution:

A is a 4×4 and $\lambda_1 = 1$is an eigenvalue of multiplicity 4; hence,

n=4, *v*=4 and n-*v*=0.

$$(\mathbf{A}\text{-}1\mathbf{I}) = \begin{bmatrix}0 & 1 & 0 & -1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

Has type 2, and

$$(\mathbf{A}\text{-}1\mathbf{I})^2 = \begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

Has type $0 = n - v$. Thus, $m = 2$, $\rho_2 = r(\mathbf{A}\text{-}1\mathbf{I}) - r(\mathbf{A}\text{-}1\mathbf{I})^2 = 2 - 0 = 2$ and

$\rho_1 = r(\mathbf{A}\text{-}1\mathbf{I})^0$ - $r(\mathbf{A} - 1\mathbf{I})^1 = 4 - 2 = 2$;

Hence a canonical basis for A will have two liges of type 2 and two liges of type 1. In order for a vector

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

To be a generalized Eigen vector of type 2, either x or z must be nonzero and w and y arbitrary. If we first choose x=1, w=y=z=0,

And then choose z=1, w=x=y=0, we obtain two liges of type 2 to be

$$\mathbf{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{y_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that we could have chosen w, x, y, z in such a manner as to generate 4 linearly independent generalized eigenvectors of type 2.

The vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{and} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Together with $\mathbf{x_2}$ and $\mathbf{y_2}$ form such a set. Thus we immediately have found a set of 4 liges corresponding to $\lambda_1 = 1$ .This set; however is not a canonical basis for **A**, since it is not composed for chains. In order to obtain a canonical basis for **A**, we use only to this vectors and form chains from them.

We obtain the two liges of type1 to be,

$$\mathbf{x_1} = (\mathbf{A} - \mathbf{I})\mathbf{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{y_1} = (\mathbf{A} - \mathbf{I})\mathbf{y_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus a canonical basis for **A** is $\{\mathbf{x_2}, \mathbf{x_1}, \mathbf{y_2}, \mathbf{y_1}\}$, which consists of the two chains$\{\mathbf{x_2}, \mathbf{x_1}\}$ and $\{\mathbf{y_2}, \mathbf{y_1}\}$each containing to vectors .

Example: 5

Find a canonical basis for **A** = $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{bmatrix}$

Solution:

The characteristic equation for **A** is $(\lambda - 3)^2(\lambda - 2)^2 = 0$; hence, λ_1=3 and $\lambda_2 = 2$ are both eigenvalues of multiplicity 2. For λ_1= 3, we find that $n - v = 2$, m = 2, ρ_2 =1 and ρ_1=1, so that a canonical basis for **A** has one generalized eigenvector of type 2 and one generalized eigenvector of type 1 corresponding to $\lambda_1 = 3$. A generalized eigenvector of type 2 is

$\mathbf{x_2}$=$\begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$.

$\mathbf{x_1}$=(**A**-3**I**) $\mathbf{x_2}$=$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$

Is a generalized eigenvector of type 1.

For λ_2 = 2, we find that $n - v$ = 2, m =1 and λ_1= 2hence there two generalized Eigen vectors of type 1 corresponding to λ_2= 2.

We obtain $\mathbf{y_1}$=$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x_1}$=$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, as the required vector. Thus a canonical basis for **A** is $\{\mathbf{x_2}, \mathbf{x_1}, \mathbf{y_1}, \mathbf{z_1}\}$

which consists of one chain corresponding two vectors $\{\mathbf{x_2}, \mathbf{x_1}\}$ and two chains containing one vector a piece$\{\mathbf{y_1}\}$ and $\{\mathbf{x_{1.}}\}$

SECTION: 3.2

JORDAN CANONICAL FORM

Every matrix is similar to an at most diagonal matrix or in more precise terminology, a matrix in Jordan canonical form. We start by defining a square matrix S_k(k represent some positive integer and has no direct bearing on the order of S_k).

$$S_k = \begin{bmatrix} \lambda_k & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_k & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_k & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}.$$

Thus, S_k is a matrix that has all of its diagonal elements equal to λ_k, all of its super diagonal elements.ie all elements directly above the diagonal elements equal to 1 and all of its other elements equal to zero.

Definition:

A square matrix A is in *Jordan canonical form* if it is a diagonal matrix or can be expressed in either one of the following two partitioned forms.

$$\begin{bmatrix} D & & & 0 \\ & S_1 & & \\ & & \ddots & \\ 0 & & & S_r \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} S_1 & & 0 \\ & \ddots & \\ 0 & & S_r \end{bmatrix}$$

Hence **D** is the diagonal matrix and $\mathbf{S_k}$(k=1,2,3,… r).

Consider the following matrices.

$$\begin{bmatrix}2&1&0&0\\0&2&0&0\\0&0&3&1\\0&0&0&3\end{bmatrix} \quad \begin{bmatrix}2&1&0&0&0\\0&2&1&0&0\\0&0&2&0&0\\0&0&0&2&1\\0&0&0&0&2\end{bmatrix} \quad \begin{bmatrix}0&0&0&0&0&0\\0&2&0&0&0&0\\0&0&2&1&0&0\\0&0&0&2&0&0\\0&0&0&0&2&1\\0&0&0&0&0&2\end{bmatrix} \quad \begin{bmatrix}2&1&0&1\\0&2&1&0\\0&0&2&1\\0&0&0&2\end{bmatrix} \quad \begin{bmatrix}2&2&0\\0&2&2\\0&0&2\end{bmatrix}$$

(a) (b) (c) (d) (e)

Matrix (a) is in Jordan canonical form, since it can be written

$\begin{bmatrix}S1&0\\0&S2\end{bmatrix}$, where $\mathbf{S_1} = \begin{bmatrix}2&1\\0&2\end{bmatrix}$ and $\mathbf{S_2} = \begin{bmatrix}3&1\\0&3\end{bmatrix}$

Matrix (b) is in Jordan canonical form, since it can be expressed

$\begin{bmatrix}S1&0\\0&S2\end{bmatrix}$, where $\mathbf{S_1} = \begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$ and $\mathbf{S_2} = \begin{bmatrix}2&1\\0&2\end{bmatrix}$

Matrix (c) is also in Jordan canonical form, since it can be also be expressed as

$\begin{bmatrix}D&0&0\\0&S1&0\\0&0&S2\end{bmatrix}$, where $\mathbf{D} = \begin{bmatrix}0&0\\0&2\end{bmatrix}$ and $\mathbf{S_1} = \mathbf{S^2} = \begin{bmatrix}2&1\\0&2\end{bmatrix}$.

Matrix (d) and (e) are not in Jordan canonical form, because of the non-zero term in (1, 4) position and the second due to the 2's on the super diagonal.

Note that, a matrix in Jordan canonical form has non-zero elements only on the main diagonal and that the elements on the super diagonal are restricted to be either zero or one. In particular a diagonal matrix is a matrix in Jordan canonical form that has all its super diagonal elements equal to zero.

Definition:

Let A be an $n \times n$ matrix. A generalized model matrix M for A is an $n \times n$ matrix whose columns considered as vectors form a canonical basis for **A** and appears in **M** according to the following rules:

(M1): All chains consisting of one vector (ie, one vector in length) appear in the first column of **M**.

(M2): All vectors of the same chain appear together in adjacent columns of **M**.

(M3): Each chain appear in **M** in order of increasing type.(ie; the generalized eigenvectors of type one appears before the generalized eigenvector of type two of the same chain, which appears before the generalized eigenvector of type three of the same chain etc.)

Example: 1

Find a generalized moden matrix **M** corresponding to the **A** given in example 5 of section 3.1

Solution:

In this example we found that a canonical basis for A has one chain of two vectors $\{\boldsymbol{x_2}, \boldsymbol{x_1}\}$and two chain of one vectors each $\{\boldsymbol{y_1}\}$and$\{\boldsymbol{z_1}\}$.

Thus the first two columns of **M** must be $\boldsymbol{y_1} \, and \, \boldsymbol{z_1}$ due to (M1) while the third and forth columns must be $\boldsymbol{x_1}$and$\boldsymbol{x_2}$ respectively due to (M3).

Hence,

$$\mathbf{M} = [\boldsymbol{y_1} \;\; \boldsymbol{z_1} \;\; \boldsymbol{x_1} \;\; \boldsymbol{x_2}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$$\mathbf{M} = [\boldsymbol{z_1}\ \ \boldsymbol{y_1}\ \ \boldsymbol{x_1}\ \ \boldsymbol{x_2}] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix}.$$

Example: 2

Find a generalized model matrix **M** corresponding to the **A** given in Example 4 of section 3.1

Solution:

In the example we found that a canonical basis for **A** has two chain consisting of two vectors a piece $\{\boldsymbol{x_2}, \boldsymbol{x_1}\} and \ \{\boldsymbol{y_2}, \boldsymbol{y_1}\}. since$ this canonical basis has no chain consisting of one vector ,(M1) does not apply.

From (M2), we assign either $\boldsymbol{x_2}$and$\boldsymbol{x_1}$ to the first two columns of M and $y_2 and y_1$to the last columns of **M** or, alternatively, $\boldsymbol{y_2}$ and $\boldsymbol{y_1}$ to the first two columns of **M** and $\boldsymbol{x_2}$and$\boldsymbol{x_1}$ to the last two columns of **M** we cannot however, define **M** =$[\boldsymbol{x_1}\ \ \boldsymbol{x_2}\ \ \boldsymbol{y_2}\ \ \boldsymbol{x_2}]$ since this alignment would split the $\{\boldsymbol{x_2}, \boldsymbol{x_1}\}$chain and violate (M2). Due to (M3), $\boldsymbol{x_1}$ must precede $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ must precede y_2.

Hence,

$$\mathbf{M} = [\boldsymbol{x_1}\ \ \boldsymbol{x_2}\ \ \boldsymbol{y_1}\ \ \boldsymbol{y_2}] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M} = [\boldsymbol{y_1}\ \ \boldsymbol{y_2}\ \ \boldsymbol{x_1}\ \ \boldsymbol{x_2}] = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Examples 1 and 2 show that **M** is not unique. The important fact, however, is that for any arbitrary $n \times n$ matrix A, there does exist at least one generalized modal matrix **M** corresponding to it. Furthermore, since the columns of **M** considered as vectors form a linearly independent set, it follows that the column rank of **M** is *n,* the rank of **M** is n, the determinant of **M** is nonzero, and **M** is invertible (that is, $\mathbf{M}^{-1}$ exists).

Now let **A** represent any $n \times n$ matrix and let **M** be a generalized modal matrix for **A**. Then, one can show that

$$\mathbf{AM} = \mathbf{MJ}$$

where **J** is a matrix in Jordan canonical form. By either premultiplying or postmultiplying the above equation by $\mathbf{M}^{-1}$ we obtain either

$$\mathbf{J} = \mathbf{M}^{-1}\mathbf{AM}$$

or

$$\mathbf{A} = \mathbf{MJM}^{-1}$$

Theorem: 1

Every $n \times n$ matrix A *is similar to a matrix in Jordan canonical form.*

Example: 3

Verify $\mathbf{J} = \boldsymbol{M^{-1}AM}$ for the example 1.

Solution:

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

We compute, $\mathbf{M}^{-1} = \begin{bmatrix} -3 & 1 & -4 & 0 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Thus, $\mathbf{M^{-1}AM} = \begin{bmatrix} -3 & 1 & -4 & 0 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \mathbf{J}$, **A** matrix in Jordan canonical form.

Example: 4

Find a matrix in Jordan canonical form that is similar to

$$\mathbf{A} = \begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}$$

Solution:

The characteristic equation of **A** is $(\lambda - 2)^3 = 0$ hence $\lambda = 2$ is an Eigen value of multiplicity three.

Following the procedures of the previous solution we find that r(A-2I)=1 and $r(\mathbf{A}\text{-}2\mathbf{I}) = 0 = n - v$. Thus $\rho_2 = 1$ and $\rho_1 = 2$ which implies that a canonical basis for **A** will contain one lige of type 2 and two liges of type 1 or equivalently one chain of two vectors $\{\mathbf{x_2}, \mathbf{x_1}\}$ and one chain of vectors $\{\mathbf{y_1}\}$.

Designating M=$[\mathbf{y_1}\ \ \mathbf{x_1}\ \ \mathbf{x_2}]$ we find that

$$\mathbf{M} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

$$\mathbf{M^{-1}} = \frac{1}{4} \begin{bmatrix} 3 & -2 & 0 \\ -1 & 2 & 0 \\ -4 & 8 & 4 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \frac{1}{4}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Chapter: 4

PROPERTIES OF JORDAN CANONICAL FORM

SECTION: 4.1

FUNCTION OF MATRICES - GENERAL CASE

We can develop a method for computing functions of non diagonalizable matrices. We begin by directing our attention to those matrices that are already in Jordan canonical form.

Consider any arbitrary n×n matrix J in the Jordan canonical form

$$J=\begin{bmatrix} D & & & & 0 \\ & S_1 & & & \\ & & S_2 & & \\ & & & \ddots & \\ 0 & & & & S_r \end{bmatrix}$$

Multiply together partitioned matrices

$$J^2=\begin{bmatrix} D & & & 0 \\ & S_1 & & \\ & & \ddots & \\ 0 & & & S_r \end{bmatrix}\begin{bmatrix} D & & & 0 \\ & S_1 & & \\ & & \ddots & \\ 0 & & & S_r \end{bmatrix}=\begin{bmatrix} D^2 & & & 0 \\ & S_1^2 & & \\ & & \ddots & \\ 0 & & & S_r^2 \end{bmatrix},$$

$$J^3=J\cdot J^2=\begin{bmatrix} D^3 & & & 0 \\ & S_1^3 & & \\ & & \ddots & \\ 0 & & & S_r^3 \end{bmatrix},$$

In general,

$$J^n=\begin{bmatrix} D^n & & & 0 \\ & S_1^n & & \\ & & \ddots & \\ 0 & & & S_r^n \end{bmatrix}. \qquad n=0, 1, 2, 3, \ldots\ldots$$

Furthermore, if $f(z)$ is well defined function for J or equivalently D, $S_1 \dots S_r$

$$f(\mathbf{J}) = \begin{bmatrix} f(\mathbf{D}) & & & 0 \\ & f(\mathbf{S}_1) & & \\ & & \ddots & \\ 0 & & & f(\mathbf{S}_r) \end{bmatrix}.$$

since $f(\mathbf{D})$ has already been determined, we only need develop a method for calculating $f(S_k)$ in order to have $f(\mathbf{J})$ determined completely . Now we have $(p+1)\times(p+1)$ matrix S_k defined by

$$\mathbf{S}_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_k & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}.$$

It can be shown that

$$f(\mathbf{S}_k) = \begin{bmatrix} f(\lambda_k) & \frac{f'(\lambda_k)}{1!} & \frac{f''(\lambda_k)}{2!} & \cdots & \frac{f^{(p)}(\lambda_k)}{p!} \\ 0 & f(\lambda_k) & \frac{f'(\lambda_k)}{1!} & \cdots & \frac{f^{(p-1)}(\lambda_k)}{(p-1)!} \\ 0 & 0 & f(\lambda_k) & \cdots & \frac{f^{(p-2)}(\lambda_k)}{(p-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f(\lambda_k) \end{bmatrix}.$$

Example: 4.1.1

Find e^{S_k} if $\mathbf{S}_k = \begin{bmatrix} 2t & 1 & 0 \\ 0 & 2t & 1 \\ 0 & 0 & 2t \end{bmatrix}$d

Solution:

In this case, we take λ_k=2t, $f(\mathbf{S_k}) = e^{s_k}$ and $f(\lambda_k) = e^{\lambda_k}$

$$e^{s_k} = f(\mathbf{S_k}) = \begin{bmatrix} f(\lambda_k) & f(\lambda_k) & f''(\lambda_k)/2 \\ 0 & f(\lambda_k) & f(\lambda_k) \\ 0 & 0 & f(\lambda_k) \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda k} & e^{\lambda k} & e^{\lambda k}/2 \\ 0 & e^{\lambda k} & e^{\lambda k} \\ 0 & 0 & e^{\lambda k} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & e^{2t} & e^{2t}/2 \\ 0 & e^{2t} & e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: 4.1.2

Find $\mathbf{J}^6$ if $\mathbf{J} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Solution:

J is the Jordan canonical form

$$\mathbf{J} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 \end{bmatrix}$$

In this case, $f(J) = J^6$.It follows that

$$\mathbf{J}^6 = \begin{bmatrix} \mathbf{D}^6 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1^6 \end{bmatrix} \qquad (1)$$

We find that,

$$\mathbf{D^6} = \begin{bmatrix} 2^6 & 0 \\ 0 & 3^6 \end{bmatrix} = \begin{bmatrix} 64 & 0 \\ 0 & 729 \end{bmatrix} \qquad (2)$$

And with $\lambda_1 = 1$

$$\mathbf{S_1^6} = \begin{bmatrix} f(\lambda_1) & \frac{f'(\lambda_1)}{1} & \frac{f''(\lambda_1)}{2} & \frac{f'''(\lambda_1)}{6} \\ 0 & f(\lambda_1) & f'(\lambda_1) & \frac{f''(\lambda_1)}{2} \\ 0 & 0 & f(\lambda_1) & f'(\lambda_1) \\ 0 & 0 & 0 & f(\lambda_1) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1^6 & 6\lambda_1^5 & 15\lambda_1^4 & 20\lambda_1^3 \\ 0 & \lambda_1^6 & 6\lambda_1^5 & 15\lambda_1^4 \\ 0 & 0 & \lambda_1^6 & 6\lambda_1^5 \\ 0 & 0 & 0 & \lambda_1^6 \end{bmatrix}$$

$$= \begin{bmatrix} (1)^6 & 6(1)^5 & 15(1)^4 & 20(1)^3 \\ 0 & (1)^6 & 6(1)^5 & 15(1)^4 \\ 0 & 0 & (1)^6 & 6(1)^5 \\ 0 & 0 & 0 & (1)^6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 15 & 20 \\ 0 & 1 & 6 & 15 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3)$$

Substituting (2) and (3) to (1), we obtain

$$\mathbf{J^6} = \begin{bmatrix} 64 & 0 & 0 & 0 & 0 & 0 \\ 0 & 729 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 & 15 & 20 \\ 0 & 0 & 0 & 1 & 6 & 15 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now let **A** be an n × n matrix. We know from the previous section that there exist a matrix **J** in Jordan canonical form and an invertible generalized modal matrix **M** such that

$\mathbf{A} = \mathbf{AJM}^{-1}$

We have,

$f(\mathbf{A}) = \mathbf{M}(\mathbf{J})\mathbf{M}^{-1}$

Providing, of course, that $f(\mathbf{A})$ is well-defined. Thus, $f(\mathbf{A})$ is obtained simply by first calculating $f(\mathbf{J})$, which can be done quite easily, then premultiplying $f(\mathbf{J})$ by $\mathbf{M}$, and finally postmultiplying this result by $\mathbf{M}^{-1}$.

Example: 4.1.3

Find $e^{\mathbf{A}}$ if

$$\mathbf{A} = \begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}$$

Solution:

From Example 4 of section: 3.2 we have that a modal matrix

for $\mathbf{A}$ is

$$\mathbf{M} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & 1 \end{bmatrix} \text{ and } \mathbf{J} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Thus, $e^{\mathbf{A}} = \mathbf{M}e^{\mathbf{J}}\mathbf{M}^{-1}$. In order to calculate $e^{\mathbf{J}}$, we note that

$$\mathbf{J} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S_1} \end{bmatrix}$$

where D is the 1×1 matrix [2] and $\mathbf{S_1}$ is the 2×2 matrix

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

we find that

$$e^{\mathbf{D}} = [e^{\mathbf{2}}],$$

$$e^{s_1} = \begin{bmatrix} e^2 & e^2 \\ 0 & e^2 \end{bmatrix}$$

and

$$e^{\mathbf{J}} = \begin{bmatrix} e^{\mathbf{D}} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{S}_1} \end{bmatrix} = \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix}.$$

Thus,

$$e^{\mathbf{A}} = = \mathbf{M}e^{\mathbf{J}}\mathbf{M}^{-1} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ -1 & 2 & 0 \\ -4 & 8 & 4 \end{bmatrix} \frac{1}{4}$$

$$= e^2 \begin{bmatrix} -1 & 4 & 2 \\ -3 & 7 & 3 \\ 4 & -8 & -3 \end{bmatrix}.$$

SECTION: 4.2

PROPERTIES OF JORDAN CANONICAL FORM

It is in general, difficult to find the Jordan canonical form of the matrix, but knowledge of certain elementary facts simplifies the task. We assume that **A** is an $n \times n$ matrix and the characteristic polynomial of **A** factors completely, say $p_A(\lambda) = (a_1 - \lambda)^{m_1} \ldots \ldots (a_s - \lambda)^{m_s}$, where $a_1, \ldots \ldots, a_s$ are distinct. Further, let the minimum polynomial of **A** be $m_A(\lambda) = (\lambda - a_1)^{n_1} \ldots\ldots (\lambda - a_{s)})^{n_s}$. Let $\boldsymbol{J}$ be Jordan canonical form of **A** and assume $J_1 \ldots J_k$ are the Jordan blocks of J.

Since $\boldsymbol{J}$ and **A** are similar they have same characteristic polynomial and since $\boldsymbol{J}$ is upper triangular, the eigenvalue of $\boldsymbol{J}$ lie on the diagonal. Therefore the following theorems are true.

Theorem: 4.2.1

*The sum of the orders of the blocks in which a_i occurs on the diagonal is m_i; ie, a_i occurs on the diagonal of **J** m_i times.*

Now let $\boldsymbol{S}$ be a non-singular matrix such that $\boldsymbol{S^{-1}AS=J}$ or $\boldsymbol{AS=SJ}$.

If $S = [X_1 \ldots X_2]$, where X_j is the j-th column of **S** then $X_1 \ldots X_n$ are linearly independent and we have

$$AS = [AX1 \quad . \quad . \quad . \quad AXn]$$

$$= SJ$$

$$= [X1 \quad . \quad . \quad . \quad Xn]\begin{bmatrix} J1 & . & . & 0 \\ . & . & & . \\ . & & . & . \\ 0 & . & . & Jk \end{bmatrix}$$

$$=[a_1X_1 \quad X_1 + a_1X_2 \ldots X_r\text{-}1 + a_1X_r \quad a_2X_r + 1 \quad X_r + 1 + a_2X_r + 2 \ldots]$$

where a_i is the eigenvalue associated with J_i and J_1 is $r \times r$. If we let $A_i = A - a_iI$ and if we equate the columns of **AS** and **SJ**, we have

$AX_1 = a_1X_1$ $\quad => A_1X_1 = 0$ $\quad A_2X_r+1 = 0$

$AX_2 = X_1+a_1X_2$ $\quad => A_1X_2 = X_1$ $\quad A_2X_r+2 = X_r+1$

. . .

. . .

. . .

$AX_r = X_r\text{-}1+a_1X_r$ $\quad => A_1X_r = X_r\text{-}1$

A basis of the above form is called a *Jordan basis*. From the above computation one sees that $X_1, X_{r+1} \ldots$ are linearly independent eigenvectors and there is one of them for each Jordan block of J.

Theorem: 4.2.2

*The no of the blocks associated with the eigenvalue a_i is equal to the number of linearly independent eigenvectors associated with a_i (There is a block in **J** for each independent eigenvectors).*

CONCLUSION

Project was done on Jordan canonical form. In this project we have seen how to find Jordan canonical form of a matrix, its properties, generalized Eigenvectors and function of matrices

We conclude from the project that Jordan canonical form is one of the important and useful concepts in linear algebra. The Jordan canonical form of a linear transformation or of a matrix encodes all of the structural information about that linear transformation or matrix.

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