GENERATING FUNCTIONS

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CERTIFICATE

This is to certify that the project report entitled "GENERATING FUNCTIONS" is a bonafide record of studies undertaken by ANN MARY PATHROSE (Reg no. 170021032398), ERICA FERNANDEZ (Reg No. 170021032409), JOSEPH ZACK JOLLY (Reg No. 170021032415) in partial fulfilment of the requirements for the award of B.Sc. Degree in Mathematics at Department of Mathematics, St. Paul's College, Kalamassery, during the academic year 2017-2020.

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DECLARATION

We, ANN MARY PATHROSE (Reg no. 170021032398), ERICA FERNANDEZ (Reg No. 170021032409), JOSEPH ZACK JOLLY (Reg No. 170021032415) hereby declare that this project entitled "GENERATING FUNCTIONS" submitted to Department of Mathematics of St. Paul's college, Kalamassery in partial requirement for the award of B.Sc Degree in Mathematics, is a work done by us under the guidance and supervision of Mr. SANEESH KUMAR V.G, Department of Mathematics, St. Paul's college, Kalamassery during the academic year 2017-2020

We also declare that this project has not been previously presented for the award of any other degree, diploma, fellowship, etc.

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CONTENT

Chapter - 1

Introduction

Generating functions is a part of combinatorics. Generating functions are a bridge between discrete Mathematics, on one hand and continuous analysis on the other. It is possible to study them solely as tools for solving discrete problems. Due to their ability to encord information about an integer sequence, generating functions are powerful tools that can be used for solving recurrence relations.

A generating function is a clothline on which we hang up a sequence of numbers for display. The origin of medieval time combinatorics were the Mathematical question on gambling that lead to the rich branch of discrete probability. Serious study of combinatorics problems is many times undertaken in order to achieve something in different field, probability begin the most important among them. This is the second reason for studying the basics of probability. At that time a gadget was being developed that could understand and also to solve a large number of counting questions and that gadget is the concept of a generating function. The powerful tool of generating function was first used by De Moivre in 1720 but was really championed by Euler in the second half of 18^{th} century. The name “generating functions” was coined by Laplace in the late 18^{th}century and was among highly used tool of Euler, The father figure of combinatorics. At a conceptual level, a generating function is a device embeds a combinatorial problem in the framework of algebra and most of the times also solves it. A generating function is a continuous function associated with a given sequence. For this reason, generating functions are very useful in analysing discrete problem involving sequence of numbers or sequence of functions.

The generating function of a sequence $\{f_n\}_{n=0}^{\infty}$is defined as f(x)=$\sum_{n=0}^{\infty} f_n \ x^n$, for |x|$<$ R and R is the radius of convergence of the series.

Chapter - 2

Partition theory of integers

2.1- Partitions and Ferrers diagrams

This chapter is aimed at studying generating functions in their application to the theory of integer partitions. Historically this area marks the beginning of modern combinatorics in the form of a very large number discoveries mainly by Euler and later by many others such as Gauss, Jacobi, Sylvester and Ramanujan.

Definitions – 2.1.1

Let "n" and "k" be positive integers. Then q(n,k) denotes the number of ordered partitions of n into k parts. Also, q(n) denotes the total number of partition of n.

Example : 2.1.1

3 = 1 + 1 + 1

= 1 + 2

= 2 + 1

= 3

Are the four ordered partition of 3.

(note that treat 1+2 and 2+1 as different ordered partitions)

Definitions – 2.1.2

A partition $n = a_1 + a_2 + \dots\dots + a_k$ into parts ($a_1, a_2, \dots. a_k \geq 1$) is said to be an unordered partition if we do not distinguish between $n=a_1+a_2+\dots.+a_k$ were $\sigma = i_1, i_2, \dots. i_k$ is a permutation of the set(k).

Example : 2.1.2

2+3+4, 3+4+2, 2+4+3, 3+2+4, 4+2+3, 4+3+2

These six ordered partitions give rise to a single unordered partition.

Definitions – 2.1.3

A partition of 'n' into 'k' parts $a_{i,}$ where i = 1,2,3…..k, will be written in the form $n=a_1+a_2+\ldots\ldots+a_k$ where $a_1 \geq a_2 \geq \ldots\ldots \geq a_k \geq 1$

Example : 2.1.3

14=2+1+2+1+5+3 is not a partition but 14= 5+3+2+2+1+1 is a partition.

Definitions – 2.1.4

For a natural number 'n' we denote the total number of partitions of 'n' by P(n) where P(0)=1 by definition.

Example : 2.1.4

3=3, 3=2+1, 3=1+1+1 are the three partition of 3. P(3)=3.

Theorem – 2.1.1

(All part partition g.f)

$P_{\text{all parts partitions}}(x) = \prod_{j=1}^{\infty}(1-x)^{-1}$

Proof-

The R.H.S is ($1+x^1+x^2+\ldots\ldots+x^x+\ldots)(1+x^2+x^4+\ldots\ldots+x^{2x}+\ldots)(1+x^3+x^6+\ldots\ldots+x^{3x}+\ldots)\ldots(1+x^j+x^{2j}+\ldots\ldots+x^{mj}+\ldots)$

The coefficient of x^n is this expression (for a positive integer n) arises from all the terms of the type

$(x^{j_1})^{m_1}(x^{j_2})^{m_2}\ldots(x^{j_r})^{m_r} = x^{m_1 j_1+m_2 j_2+\cdots+m_r j_r}$

Where $m_1 j_1$+$m_2 j_2$+…+$m_r j_r$ = *n* and $j_1 \geq j_2 \geq \ldots \geq j_r$

($without\ any\ loss$ generality) and indeed this expression correspondes to the partition

n=j_1+j_1+…+j_1+ j_2+ j_2 + … +j_2+……+ j_r+j_r+…+j_r

⬇ ⬇ ⬇

$m_1 m_2 m_r$

For example,

x^{50}=$x^{4\times7}x^{1\times5}x^{3\times4}x^{2\times2}x^1$, corresponds to

50 = 7 + 7 + 7 + 7 + 5 + 4 + 4 + 4 + 2 + 2 + 1

This shows that on the R.H.S we are actually counting all the partitions of n when we consider the coefficient ofx^n.

As an example consider the coefficient of x^4 on the R.H.S. We should only be looking at $f_1(x)f_2(x)f_3(x)f_4(x)$ because when j≥5, the non-trivial terms in $f_j(x)$ involve at least 5-th power of x. We begin by choosing x^4 from $f_4(x)$ (and 1 from the rest). This is the only way $f_4(x)$ can contribute to our stipulation. Now consider $f_1(x)f_2(x)f_3(x)$and we may take x^3 from $f_3(x)$ and then must take *x* from $f_1(x)$. Again, this is the only way $f_3(x)$ can contribute. Next consider $f_1(x)f_2(x)$. Here we can choose x^4 from $f_2(x)$ (and 1 from $f_1(x)$). We can also choose x^2 from $f_2(x)$ and x^2 from $f_1(x)$. Finally we can choose x^4 from $f_1(x)$ giving a total of 5 partitions. We have not really simplified the problem; in fact these 5 partitions are

π_1: 4 = 4

π_2: 4 = 3 + 1

π_3: 4 = 2 + 2

π_4 : 4 = $2 + 1 + 1$

π_5 : 4 = 1+1+1+1

A slightly different ways of doing the same thing , which uses more algebra , but not very useful in this case is the following.

$$f_1(x)f_2(x)f_3(x)f_4(x) = \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^4}$$

$$= \frac{1+x}{(1-x^2)^2}\frac{1}{1-x^4}\frac{1}{1-x^3}$$

$$= \frac{(1+x)\left(1+x^2\right)^2}{(1-x^4)^3}\frac{1}{1-x^3}$$

$$=(1+x)(1+x^2)^2 \times \sum_{k=0}^{\infty}\binom{k+2}{k} x^{4k}$$

$$=(1+x+2x^2+2x^3+x^4+x^5) \times \sum_{k=0}^{\infty}\binom{k+2}{k} x^{4k} \times \sum_{r=0}^{\infty} x^{3r}$$

The coefficient of x^4 can now be read off by making cases (k,r)=(1,0) when the number is $\binom{3}{2}$=3 and (k,r)=(0,1) when the number is 1 and finally when (k,r)=(0,0) when the number is 1 giving the same answer.

As a convention ,we use greek letters α, β, π etc to denote partitions. If π is the partition n=a_1+a_2+...+a_k, then we write $\pi = \left(a_{1,}a_{2,\ldots}a_k\right)$identifying π with the sequence of parts written in a monotone decreasing manner and the integer represented by π is uniquely determined.

Hence the proof.

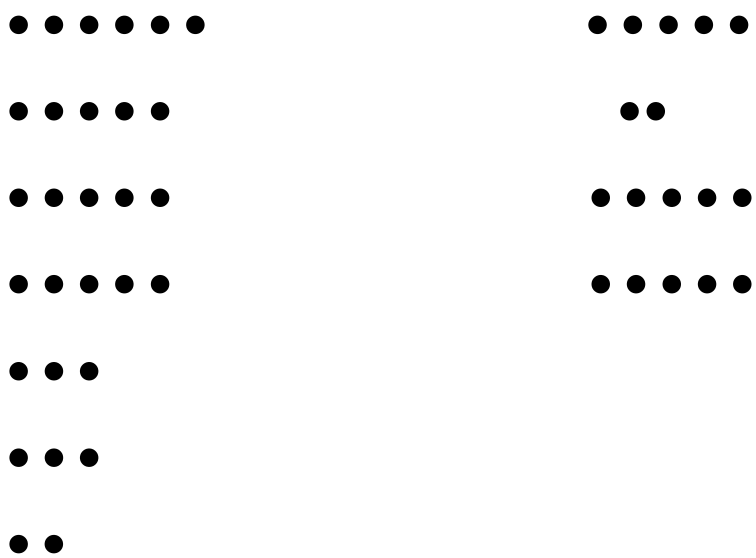
Ferrers Diagram

Definition – 2.1.5

A Ferrers diagram F is an array that consists of certain number of dots, say n $=a_1+a_2+\cdots+a_k$ arranged in some k rows with i-th row containing exactly a_i dots. Here $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ and the dots are arranged regularly with no gaps between two consecutive dots in a row or a column such that the following conditions is satisfied: The rows of F are left justified. That is, the first dot in every row begins at the same column.

Example of a ferrers diagram : 2.1.5

In figure (i), the first diagram is an example of a Ferrers diagram while the second is not. The second is not a Ferrers diagram because it fails to be one on all the three counts: the second row is not left justified, the last row is bigger than the one before it and we also have a gap in the second row.



Ferrers diagram figure (i) Not a Ferrers diagram

Example

If π = (4 , 3 , 3 , 1 , 1) is a partition (of the number 12) then its ferrers is given by

● ● ● ●

● ● ●

● ● ●

●

●

Example

Ferrers diagram of the five partitions of 4

●	● ●	● ●	● ● ●	● ● ● ●
●	●	● ●	●	F5
●	●	F3	F4	
●	F2			
F1				

Definition – 2.1.6

A partition π is called a self conjugate partition if it is the same as its conjugate that is $\pi'=\pi$

Example : 2.1.6

The two partitions π_1 = (3,1,1) and π_2 = (3,2,1) are both self-conjugate partitions of numbers 5 and 6 respectively and have Ferrers diagrams given in figure(ii)

Figure(ii)

Note

Notice that if $\pi = \left(a_{1,}a_{2,}a_{3,} \dots , a_k\right)$ and $\sigma = (b_{1,}b_2, b_{3,\dots,}b_m)$ are conjugate partitions , then b_1= k and m=a_1.Thus ,the numbers of parts in σ equals the size of the largest part of π and the size of the largest part of the σ is the same as the number of parts of π.

Theorem – 2.1.2

Let n and k be natural numbers. Then the number of partitions of n into k parts is equal to the number of partitions of n in which the largest part is k,also the number of partitions of n into at the most k parts is equal to the number of partitions of n in which each part is less than or equal to k.

Proof

Let S denote the set of all the partitions of n. In both the cases, we can construct two subsets of S say T and T^{l} such that $T^{l} = \left\{\pi^{1} : \pi \epsilon T\right\}$. In the first case, T consists of partitions for which the largest part is k. Then the bijection between T and T^{l} proves the required result. For the second part, note that the partitions of n in which each part is $\leq$k are same as those in which the largest part is $\leq$k.

Example : 2.1.7

There are three partitions of 6 in which the largest part is 3 and these are

$$6 = 3 + 3$$
$$= 3 + 2 + 1$$
$$= 3 + 1 + 1 + 1$$

and by conjugation , there are also three partitions of 6 in which we have exactly three parts

$$6 = 2 + 2 + 2$$
$$= 3 + 2 + 1$$
$$= 4 + 1 + 1$$

Similarly we have three partitions of 5 in which the numbers of parts is ≤ 2 and these are:

$$5 = 2 + 2 + 1$$
$$= 2 + 1 + 1 + 1$$
$$= 1 + 1 + 1 + 1 + 1$$

And also we have three partitions of 5 in which the number of parts is at the most 2.

$$5 = 3 + 2$$
$$= 4 + 1$$
$$= 5$$

Definition - 2.1.7

A partition π=$\left(a_{1,}a_{2,} \dots, a_k\right)$ is called a distinct part partition if $a_1 > a_2 >$……$> a_k \geq 1$. The number of distinct part partitions of n is denoted by d(n).

Example : 2.1.8

The number of distinct part partitions of 6 is four and these are:

6	=	6
	=	5 + 1
	=	4 + 2
	=	3 + 2 + 1

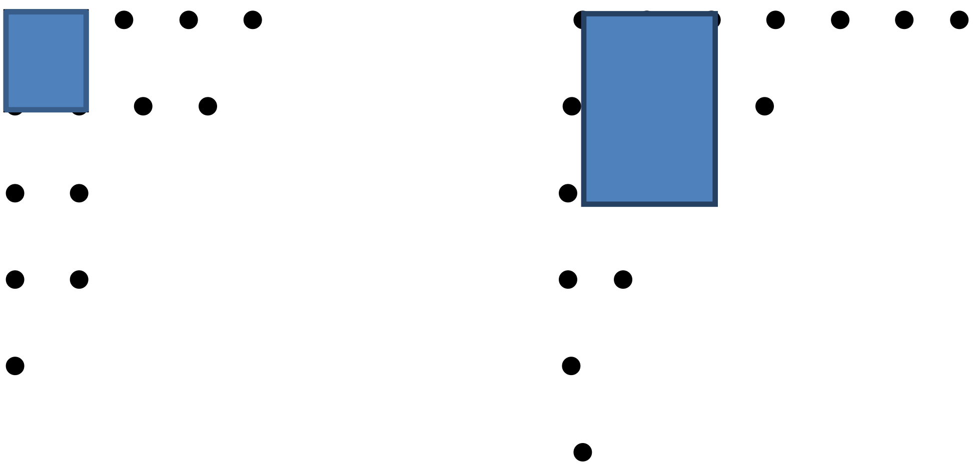
Let 0(n) denote the number of partitions of n into parts each of which is odd.

2.2- Durfee Squares and self conjugate partitions

Given a Ferrers diagram F, the Durfee squares D_m of F is a maximal m×m square of dots in the given diagram that includes consecutive dots both from left and right and top and bottom that begins at the left top corner dot.

Example : 2.2.1

The Durfee squares in the following two diagrams (in figure iii) F_1 and F_2 are D_2 and D_3 respectively.



Figure(iii)

Consider the two Ferrers diagram F_1 and F_2 in figure(iii). In the case of F_1, we have m=2 and the partition to the right of the Durfee squares is π_R= (3,2) while the one below the Durfee square is

π_B= (2,2,1).

In the case of F_2, we have m=3 and the partition to the right of the Durfee square is π_B= (2,2,1). This procedure is reversible. That is given m$\geq$1, given a partition σ with at the most m parts and given a partition T with the largest part $\leq$m, we can stick σ to the right of D_m and T under $D_m$to get a partition π that has D_m as its Durfee squares.

Chapter – 3

Recurrence Relations

3.1– Introduction

A Recurrence relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous terms.

The simplest form of a recurrence relation is the case where the next term depends only on the immediately previous term. If we denote the n^{th} term in the sequence by x_n, such a recurrence relation is of the form $x_{n+1} = f(x_n)$ for some function f.

A recurrence relation can also be a higher order, where the term x_{n+1} could depend not only on the previous term x_n but also on earlier terms Such as x_{n-1} , x_{n-2} etc. A second order recurrence relation depends on x_n and x_{n-1} and is of the form $x_{n+1} = f(x_n, x_{n-1})$ for some function f.

3.2 The case of repeated roots

Lemma 3.2.1

Let q a non - zero complex number and let f(x) be a non zero polynomial. Let r be a positive integer. Recursively define : $f_1(x) = f(x)$, $f_2(x) = xf'_1(x) \ldots f_r(x) = x\, f_{r-1}(x)$.Then the following statements are equivalent.

a) q is a root of f(x) with multiplicity $\geq r$
b) $f_i(q) = 0$ for all i=1,2 ...r

Proof

Let (a) hold if r = 1, then there is nothing to prove.

Let r≥2 then q is zero of multiplicity ≥r-1 of f'(x) and of f_2 (x) = xf'(x). Since f_3,f_4…f_r are defined in the same manner from f_2(r),it follows by induction that f_i(q) = 0 for all i=1,2…r.

Let (b) hold with r ≥2 again since f_2(q) = 0 and f_3 …f_r are defined from f_2 ,it follows by induction that q is a root of f_2(x), with multiplicity ≥ r-1. But q≠0 and hence is a root of f'(x), with multiplicity ≥r-1, and therefore that of f(x) with multiplicity ≥ r.

Lemma 3.2.2.

Let q be a nonzero with multiplicity r ≥ 2 and let j=0,1…r-1.then $H(n) = n^j q^n$

Proof

Let J ≥ 1. Let $f(x) = x^{n-k} p(x)$. Then q is a root of f(x) with multiplicity r. Defining f_j(x) recursively in the same manner as in the above lemma. We see that,

$$f(x)=x^n-a_1x^{n-1}-a_2x^{n-2}-\ldots-a_{k-1}x^{n-k+1}-a_kx^{n-k}$$

$$f_3(x)=n^3x^n-a_1(n-1)^3 x^{n-1}-a_2(n-2)^3 x^{n-2}-\ldots-a_{k-1}(n-k+1)^3 x^{n-k+1}-a_k(n-k)^3 x^{n-k}$$

using above lemma we get $f_3(q)=0$

$$\Longrightarrow n^j q^n=a_1 (n-1)^j q^{n-1} +a_2 (n-2)^j q^{n-2} +\ldots+a_{k-1}(n-k+1)^j q^{n-k+1}+a_k(n-k)^j q^{n-k}$$

Showing that $H(n)=n^j q^n$ is a solution.

Theorem 3.2.1

Let the characteristic polynomial p(x) factorize into p(x) = $(x-q_1)^{r1}$ $(x-q_2)^{r2}$ … $(x-q_n)^{rm}$ where q_1, $q_2 \ldots q_m$ are distinct non zero complex numbers and $r_1, r_2 \ldots r_m$ are positive integers then

$h(n)=\sum_{i=1}^{m}\{\sum_{j=1}^{n} c_{ij}\, n^{j-1}\}q_1^{\,n}$

Where c_{ij} with i=1,2…m and j=1,2…r_i and k are arbitrary constants is a general solution of the recurrence relation.

Proof

Since the solution space is closed under linear combinations we see that h(n) is a solution using the previous lemma. It is therefore sufficient to prove that the given arbitrary numbers $b_0, b_1 \ldots b_{k-1}$,we can find constants c_{ij}'s such that the following k linear equations are satisfied.

Consider the equation determined by b_0:-

$C_1.1 +0 +\ldots+0+C_2, 1+0+\ldots+C_m,1+0+\ldots0=B_0$

For this equation the coefficient of C_i,1 is 1 while that of C_{ij} is zero. If J≥2 (since $0^j=0$).

Next we have

$C_1q_1+\ldots+c_1r_1q_1+\ldots c_{m1}q_m+\ldots c_{m1}r_mq_m=b_1$

$C_1q_1^{\,2}+c_{12}2q_1^{\,2}+\ldots c_1r_12^{r_1-1}q_1^{\,2}+\ldots+c_{m1}q_m+\ldots+c_mr_m2^{r_m-1}q_m^{\,2}=b_2$

So on , since h(s) = b_r we have in general

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\ 8^{j-1}q_1 = b_i$$

Here $0_m = 0$ if u≥1 and 0^0=1,we have a matrix equation $0\bar{c}=\bar{b}$ where $\bar{b}$ is the column vertex

$\bar{b}$=(b_0,b_1….b_{k-1})^i and, $\bar{c}$=(c_1,1…c_1,r_1…c_{m-1}…c_m,r_m)

And the coefficient matrix Q is a K×K matrix given by Q ={a_3,(i,j)} where i=1,2…m and j=1,2…r_i for a given i specifically as $(i,j) = \delta^{j-1} q_i^2$ To simplify things we observe that Q =[Q_1,Q_2…Q_m] where Q is a K × r matrix corresponding to the i^{th} root of q. For the sake of simplicity, if we write for q for Q_i then the first column of the matrix Q_i is $[1,q,q^2…q^{k-1}]^t$ while for J≥2 the J^{th} column for the matrix is :

$$(0,q,2^{j-1}\, q^2,3^{j-1}\, q^3…(k-1)^{j-1},\, q^{k-1})^t$$

Suppose we prove that q is a non singular. Then clearly $\tau = \theta^{-1}\, \bar{b}$ gives a unique solution and prove that the constants are indeed uniquely determined. Note that the non singularity of Q is equivalent to the assection that the row null – space of Q has dimensions zero.

3.2-Difference tables and sums of polynomials.

Defintion 3.2.1

For a real value function f(x), defined on R we define Δf(x) = f(x+1)-f(x) called the forward difference of f(x)

Observe that Δf(x) is also a Real valued function defined on R. Differences are used in numerical analysis because they approximate the “differential” of a function.

eg. $f(x) = x^2+2x+3$ then,
$\Delta f(x)=[(x+1)^2 +2(x+1)+3]-[x^2+2x+3]=2x+3$

Definition 3.2.2

The ordered set (f(0), Δf(0), Δ^2 f(0)…) consisting of the left most entries in the difference table of f is called the left edge of f.

Theorem 3.3.3

Let f be a polynomial of degree 'n'. Then,

(a) The left edge of f is (f(0), Δf(0), Δ^2f(0)…Δ^n f(0), 0,0…) and we can conveniently write this as (f(0), Δf(0), Δ^2f(0)…Δ^nf(0)) after deleting the 0's at the end.

(b) The left edge of f determines f uniquely

(c) Let n be a positive integer and let $f_n(x) = \binom{x}{n}$ then the left edge of $f_n(x)$ is the sequence (0,0…q_i) where i is at the n^{th} place.

Proof

The difference table of f is the same as that of Δf with the top or the 0^{th} row (consisting of f(0), f(1), f(2) …) added to it in particular the left edge of f is that as same that of Δf with the entry f(0) augmented to the left.We know that Δf is a polynomial of degree n-1 with the left edge (Δf(0), Δ^2f(0)…Δ^n f(0)) (using induction) and hence the left edge of f is (f(0), Δf(0), Δ^2 f(0)…Δ^n f(0)) as desired. For (b) we use induction again to see that left edge (Δf(0), Δ^2 f(0)…Δ^n f(0)) of Δf determines it uniquely and in particular, Δf(1), Δf(2)… are uniquely determined. Since f(j+1) = f(j) + Δf(j) we see that all of f(0),F(1),F(2)… are recursively and uniquely determined. For (c) use pascal identity

$$\Delta f_n(x) = \binom{x+1}{n} - \binom{x}{n} = \binom{x}{n-1} = f_{n-1}(x)$$

By induction the left edge of $f_{n-1}(x)$ is (0,0,…0,1) with 1 at the $(n-1)^{th}$ place and hence argumenting f(0)=0 to this on the left we get (0,0,…0),1 at the n^{th} place with 1.

Example 3.3.1

Let p(x)=x^4=$3x^3$-$7x^2$=5x-2.the difference table is,

2		4		24		116		358		852
	2		20		92		243		494	
		18		72		150		252		
			54		78		102			
				24		24				
					0					

The last entry (at the bottom) was not really necessary.Since we already know that Δ^5f=0.

The key to finding a sum of the form $\sum_{x=0}^{m} p(x)$ is an identity proved as generalization of pascal identity i.e;

$$\sum_{x=0}^{m} f_n(x) = \sum_{x=0}^{m}\binom{x}{n} = \binom{m=1}{n=1}$$

To find $\sum_{x=0}^{m} p(x)$ we express p(x) of degree n as a linear combination of the polynomials f_r(x) and obtain the requires sum to facilitate this, recall the identity on stirling numbers of second kind i.e;

X^n=$\sum_{k=1}^{n} s(n,k)[x]_k$ valid for all positive integers n. from this we get x^n=$\sum_{k=1}^{n} s(n,k)k!\binom{x}{k}$ and this can be used to obtain p(x) in terms of f_k(x) and the use of generlised pascal identity will then obtain the sum $\sum_{x=0}^{m} p(x)$ as a closed polynomial expression in the variable m.

3.4- Other types of recurrence realtions

No general method is known for solving a recurrence relation which is not of the type discussed in the earlier sections. Inspite of that ,we discuss some other methods of recurrence relations that are not linear, homogenous with constant coefficients. An example of this is the towers of Hanoi problem. We assume that the recurrence relation has the form

$$H(n)=f(H(n-1),H(n-2)\dots H(n-k))+G(n)$$

Where f is the linear and homogeneous function of its variables with constant coefficients G(n) the non homogeneous part ,is a function of variable n. We also assume that G(n) has some nice form such as a polynomial in "n" or experimental function in n.

If G(n)=0

Then we can solve the part

$$H(n)=f(H(n-1)\dots H(n-k))$$

Using the earlier theory upto determination of coefficients.Then we set to find particular solution for which we have the *THUMB RULES.*

Assuming that G(n) is a sum,

$$G_1(n)+G_2(n)+\dots+G_r(n)$$

We try to obtain a particular solution in each case,

$H(n)=f(H(n-1)\dots H(n-k))+G_i(n)$ for this we equate the coefficients on both sides (treating the equation as an identity in n).that gives us r particular solutions. These are often combined with the general solution(obtained using the homogenous part and finally the general solution is found by determination of constant coefficients using the initial conditions. It remains to describe the thumb rules. Let

$G_i(n)=f(n)$ depending upon what f(n) is, the form of the particular solution is described in the following table.

Here d and B are numbers that are to be determined by the given conditions.

f(n)	**Form of particular solution**
d	B
db	$B_1n=B_0$
db^2	$B_2n^2+B_1n=B_0$
d^n	Bd^n

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Chapter-4

Applications And Conclusions

Applications

Generating functions have useful applications in many fields of study .In this paper the generating functions will be introduced and their applications is used in combinatorial problems and recurrence equations. Now we shall discuss an application of generating functions to linear recurrence problems.

1 .From Recursion to Algebra

Generating functions can be used to solve a linear recurrence problem. The problem of linear recurrence is to find the values of a sequence$\{u_n\}$ satisfying $\sum_{k=0}^{m} c_k\ u_{n+k}$= V ,for some constant V and any integer n≥o, given the initial values $u_{0\,,}u_1$,….u_{m-1} where both $c_o$and c_m non zero. In order to solve this recurrence problem , we use the following property of generating functions.

- If $\{u_n\}_{n=0}^{\infty}$ is a sequence with generating function u[x] and k is a positive integer , then $\sum_{n=0}^{\infty} u_{n+k}\ x^n$ = $\frac{1}{x^k}\left(u(x) - \sum_{j=0}^{k-1} u_j x^j\right)$

2 . Applications to Combinatorial Problems

Many combinational problems can be solved with the aid of generating functions . In particular, let’s consider the problem of finding the number of partitions of a natural number

Definition 4.1

A partition of a natural number n is a way to write n as a sum of natural numbers, without regard to the ordering of the numbers.

Example 4.1

1+1+3+1 is a partition of 6.

With this definition , the generating function of the number of partitions of n has a simple form.

Definition 4.2

The coefficient of x^n is equal to the number of partitions of n.

Another important combinatorial problem that can be easily solved with generating functions is Catalan's problem .

Example 4.2 - Catalan's Problem

Given a product of n letters, how many ways can we calculate the product by multiplying two factors at a time , keeping the order fixed ?

As an example , for n= 3 , there are two ways: $(a_1\ a_2\ )\ a_3$ and $a_1\ (a_2\ a_3)$.

This problem was solved by Catalan in 1838 and the Catalan numbers are conventionally defined as C_n=k_{n+1}, for n≥0.

3.Useful Trick to find a Generating Function

In **2,** we saw that we can easily find the generating function of a sequence if that sequence is defined through a linear reccurrence. However, in some cases, we may not have a linear recurrence, such as in the Catalan's problem in **2**. For some sequences without a linear recurrence, it is possible to obtain the generating function using a convolution property. In fact, we have actually used this property to solve the Catalan's problem.

Definition 4.3

A convolution of two sequence $\{f_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ is another sequence denoted by $\{(f * g)_n\}_{n=0}^{\infty}$ with $(f * g)_n$=$\sum_{k=0}^{\infty} f_k\, g_{n-k}$.

Conclusion

We have discussed some basic applications of generating functions, as a method to solve a linear recurrence or combinatorial problems. However, there are certainly many more aspects in the subject that are not discussed here.

Reference

1. Combinatorial – Techniques – Sharad.S.Sane