DIGRAPHS

PROJECT SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENT FOR

THE BACHELOR OF SCIENCE DEGREE IN

MATHEMATICS

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CERTIFICATE

This is to certify that the project report entitled "DIGRAPHS" is a bonafide record of studies undertaken by FATHIMA NASREEN V.K (Reg no.170021032410), ANN MINU THOMAS (Reg No. 170021032399), SANDRA V.P (Reg No. 170021032426) in partial fulfilment of the requirements for the award of B.Sc. Degree in Mathematics at Department of Mathematics, St. Paul's College, Kalamassery, during the academic year 2017-2020.

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DECLARATION

We, FATHIMA NASREEN V.K (Reg no. 170021032410), ANN MINU THOMAS (Reg No. 170021032399), SANDRA V.P (Reg No. 170021032426) hereby declare that this project entitled "DIGRAPHS" submitted to Department of Mathematics of St. Paul's college, Kalamassery in partial requirement for the award of B.Sc Degree in Mathematics, is a work done by us under the guidance and supervision of Dr MANJU K. MENON, Department of Mathematics, St. Paul's college, Kalamassery during the academic year 2017-2020.

We also declare that this project has not been previously presented for the award of any other degree, diploma, fellowship, etc.

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DIGRAPHS

CHAPTER 1

INTRODUCTION

In mathematics, Graph Theory is the study of graph, which are mathematical structures used to model pair wise relations between objects. A graph is made up of set of objects called vertices which are connected by edges or arcs. A graph may be undirected, meaning that there is distinction between the two vertices associated with each edges, or its edges may be directed from one vertex to another graphs can be used to represent almost any problem involving discrete arrangements of objects, where concern is not with the internal properties of these objects but with the relationship among them.

Graph Theory was born in 1736 with Euler's paper in which he solved Koningsberg bridge problem. After discovering of a subject which may be based on graphs, it took 200 years for the first book on Graph Theory to be published.

If $V(G)\times V(G)$ is considered as a set of ordered pairs, where V be the vertex set of graph G, then the graph G is called a directed graph or digraph for short. We investigate most of the important and fundamental features of directed graphs. Applications of digraphs are virtually unlimited.

In the first chapter, we study digraph and its different types. Also introduced the topic relationship between binary relations and digraphs. Also introduce the topic tournaments.

In the second chapter, we introduce Euler digraph and matrix representation of digraphs. Third chapter is the concluding chapter of the project in which we discuss some of its applications.

CHAPTER-2

INTRODUCTION TO DIGRAPHS

2.1 DIGRAPHS

A Directed graph or digraph D= (V, A) consists of two finite sets V, the vertex set, a non empty set of elements called the vertices of D and A, the arc set, a possibly empty set of elements called the arcs of D, such that each arc 'a' in A is assigned an ordered pair of vertices (u, v).

As in case of undirected graphs, a vertex is represented by a point and an arc by a line segment between u and v with an arrow directed from u to v. Thus, for example fig 1.1 represents directed graph D with vertex set V= {v1, v2, v3, v4} and arc set A={a1, a2, a3, a4, a5, a6, a7}.

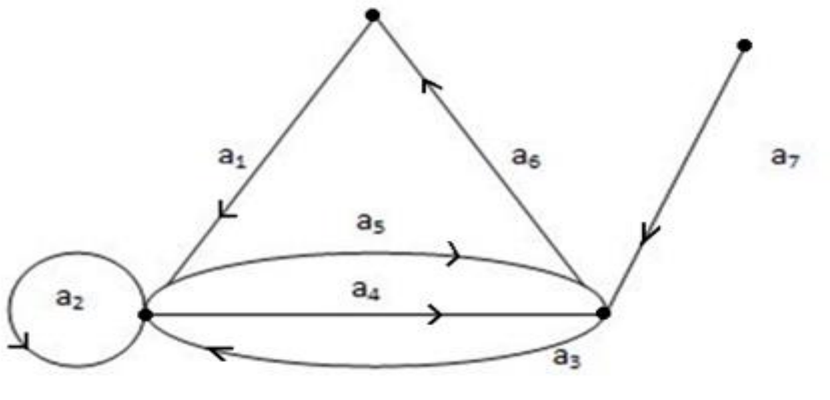


fig. 1.1

If a is an arc, in the directed graph D, with associated ordered pair of vertices (u, v) then a is said to join u to v, u is called the origin or the initial vertex or the tail of a, and v is called the terminus or the terminal vertex or head of a.

An Arc for which initial and terminal vertices are the same forms a self loop and two directed graph edges are said to be parallel if they are mapped into the same ordered pair of the vertices.

Given a digraph D, we can obtain a graph G from D by "removing all arrows" from the arcs. This graph G has the same vertex set as D and corresponding to each arc a in D with associated ordered pair of vertices (u, v), there is an edge e in G with associated pair (u, v), the G is called **underlying graph** of D.

2.2 TYPES OF DIGRAPHS

Digraphs come in many varieties. In fact due to the choice of assigning a direction to each arc, directed graphs have more varieties than undirected one.

1.Simple Digraphs

Digraph that has no self loop or parallel edges is called a simple digraph.

2. Asymmetric Digraphs

Digraphs that have at most one directed edge between a pair of vertices but are allowed to have self-loops are called asymmetric or anti-symmetric digraphs.

3. Symmetric Digraphs

Digraphs in which for every arc (a, b)there is also an arc (b, a). Digraph that is both symmetric and simple is called a **simple symmetric** graph. Similarly a digraph that is both simple and asymmetric is **simple asymmetric.**

4. Complete Digraphs

A complete undirected graph was defined as a simple graph in which every vertex is joined to every other vertex exactly by one arc.

a)Complete Symmetric Digraph

A complete symmetric digraph is a simple digraph in which there is exactly one arc directed from every vertex to every other vertex.

b) Complete Asymmetric Digraph

A complete asymmetric digraph is an asymmetric digraph in which there is exactly one arc between every pair of vertices. A complete asymmetric digraph of n vertices contains $\frac{n(n-1)}{2}$arcs, but a complete symmetric digraph of n vertices contain n(n-1) arcs.

2.3 DIGRAPHS AND BINARY RELATIONS

The theory of graphs and the calculus of binary relations are closely related. In a set of objects X, where X= $\{x_{1,}x_2, ...\}$, a binary relation R between pairs (x_i, x_j) may exist. In which case, we write $x_i R x_j$ and say that x_i has relation R tox_j.

Relation R may for instance be "is parallel to", "is congruent to" etc. A digraph is the most natural way of representing a binary relation on a set X. Each x_i ∈X is represented by a vertex x_i. If x_i has a specific relation R to x_j, an arc is drawn from vertex x_i to x_j, for every pair (x_i, x_j). Clearly, every binary relation on a finite set can be represented by a digraph without parallel edges. Conversely, every digraph without parallel edges defines a binary relation on a set of its vertices.

1.Reflexive Relation

Let R define a relation on a non empty set X. If R relates every element of X to itself, the relation R is said to be reflexive. The digraph of a reflexive relation will have self loop at every vertex. Such a digraph representing a reflexive binary relation on its vertex set may be called a reflexive digraph. A digraph in which no vertex has self-loop is called an irreflexive digraph.

2. Symmetric Relation

A relation R is said to be symmetric if for allx_i; x_j ∈ X, $x_iRx_j$implies x_jRx_i. The digraph of a symmetric relation is a symmetric digraph because for every arc from vertex $x_i$to $x_j$there is an arc from x_j tox_i.

3.Transitive Relation

A relation R is said to be transitive if for any three elements x_i, x_j andx_k belongs to X, x_iRx_j and x_jRx_k always imply x_iRx_k. A digraph representing a transitive relation is called a transitive directed graph.

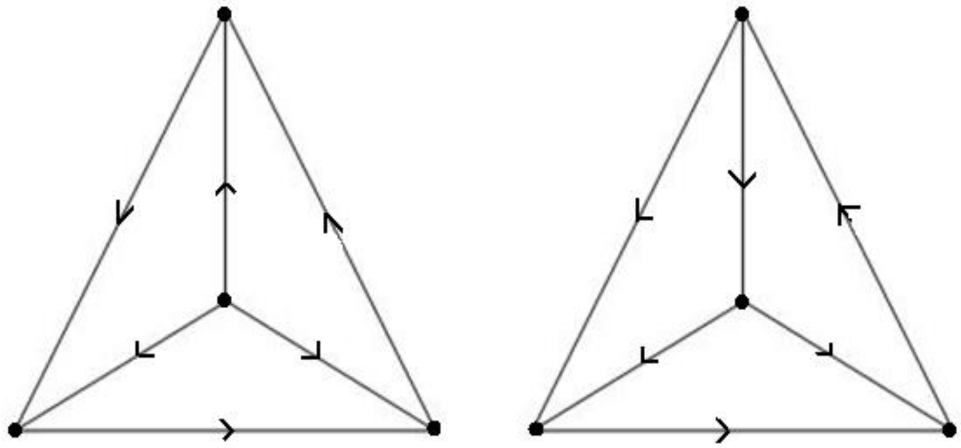
4.Equivalence Relation

A binary relation is called equivalence relation if it is reflexive, symmetric and transitive. The graph representing equivalence relation is called an equivalence graph.

<u>Isomorphic Digraphs</u>

Two digraphs are said to be isomorphic if their underlying graphs are isomorphic and the direction of the corresponding arcs are same.

Two non isomorphic digraphs are shown.



Let D =(V, A) be a digraph. A digraph H = (U, B)is a subdigraph of D whenever U ⊆V and B ⊆A . If U = V, the subdigraph is said to be Spanning.

Complement of a digraph

The complement $\overline{D}$ = ($V, \bar{A}$) of the digraph D = (V, A) has the vertex set V and $a \in \bar{A}$ if and only if a not belongs to A. That is, $\overline{D}$ is the relative complement of D in K_n where $|V|= n$.

Converse of a digraph

The converse D' = (V, A')of the digraph D = (V, A) has vertex set V and $a= uv \in A'$ if andonly if a' = vu ∈ A i.e., A' is obtained by reversing the direction of each arc of D. Clearly (D')' = D'' = D.

A digraph D is self complementary if $D \cong \overline{D}$ and D is said to be self converse if D ≅ D'. A digraph D is said to be self-dual if $D \cong \overline{D} \cong D'$.

2.4 DIRECTED PATHS AND CONNECTEDNESS

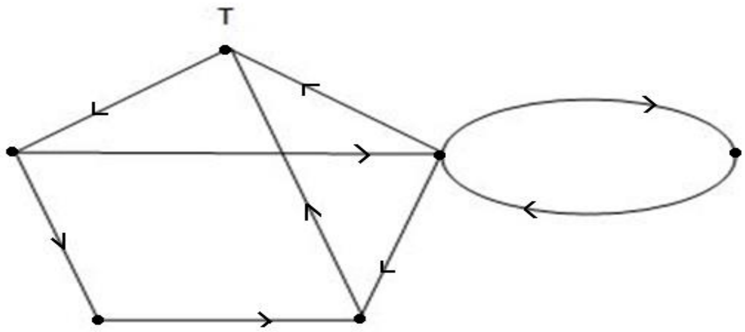
A directed walk in a digraph D = (V, A) is a sequence $v_0 a_1 v_1 a_2 \ldots a_k v_k$, where $v_i \in V$ and $a_i \in A$ are such that $a_i = v_{i-1} v_i$for 1≤i≤k, no arc being repeated.

A directed path is an open walk in which no vertex is repeated. A directed cycle is a closed walk in which no vertex is repeated.

A semiwalk is a sequence is a sequence $v_0 a_1 v_1 a_2 \ldots a_k v_k$ with $v_i \in V$ and $a_i \in A$ such that either $a_i = v_{i-1} v_i$ or $a_i v_i v_{i-1}$ and no arc is repeated. The length of semiwalk is k.

A digraph D is said to be weakly connected or connected if its underlying graph is connected and it is said to be strongly connected if for any pair of vertices u and v in D, there is a directed path from u to v. That is, given any pair of vertices in D, each is reachable from the other.

Example for strongly connected digraph

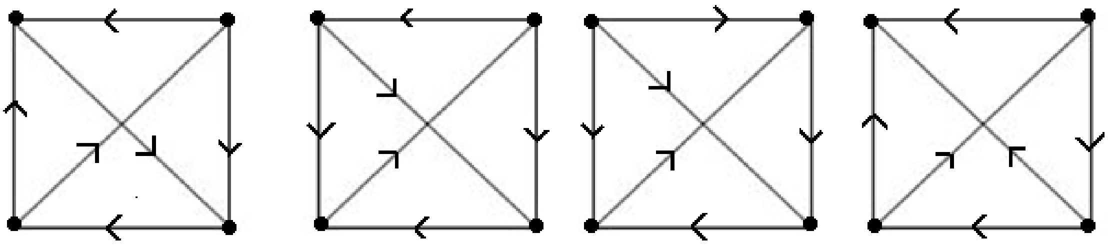


Given a graph G we can obtain a digraph from G by specifying for each edge in G an order to its end vertices. Such a digraph D is called an orientation of G.

2.5 TOURNAMENTS

A tournament is an orientation of a complete graph. Therefore, in a tournament each pair of distinct vertices v_i and v_j is joined by one and only one of the orientation arcs (v_i, v_j) or (v_j, v_i). If the arc (v_i, v_j) is in T, then we say v_i dominates v_j and is denoted by $v_i \rightarrow v_j$.

Example for tournaments on four vertices



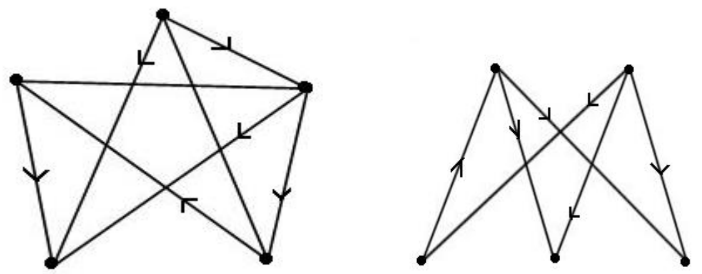
The reason for the name "tournament" is that the digraph can be used to record the results of the games in a round robin tournament in any game in which draws are not allowed, such as tennis. The arc from a to b then indicates that a has beaten b.

Definition

A triple in a tournament T is the sub digraph induced by any three vertices. A triple (u, v, w) in T is said to be transitive if whenever $(u, v) \in A(T)$ and $(v, w) \in A(T)$, then $(u, w) \in A(T)$. That is, whenever $u \rightarrow v$ and $v \rightarrow w$, then $u \rightarrow w$.

Definition

A bipartite tournament is an orientation of a complete bipartite graph. A k-partite tournament is an orientation of a complete k-partite graph. Figure below displays a bipartite and a tripartite tournament.

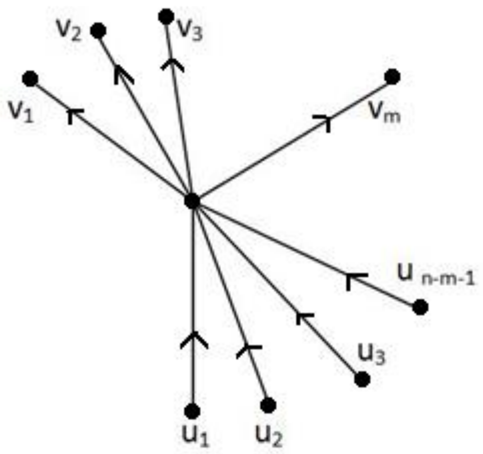


THEOREM

If *v* is a vertex having maximum out degree in the tournament T, then for every vertex *w* of T there is a directed path from *v* to *w* of length at most two.

Proof

Let T be a tournament with n vertices and let *v* be a vertex of maximum out degree in T. Let $d^{+}(v) = m$ and let $v_1, v_2, \ldots v_m$ be the vertices in T such that there are arcs from v to v_i , $1 \leq i \leq m$. Since T is a tournament, there are arcs from the remaining $n - m - 1$ vertices, say $u_1, u_2, \ldots, u_{n-m-1}$ to v. That is, there are arcs from u_j to v, $1 \leq j \leq n - m - 1$.



Then for each i, $1 \leq i \leq m$, the arc from v to v_i gives a directed path of length 1 from v to v_i .

We now show that there is a directed path of length 2 from v to u_j from each j, $1 \leq j \leq n\text{-}m\text{-}1$.

Given such a vertex u_j, if there is an arc from v_i to u_j for some i, then vv_iu_j is a directed path of length 2 from v to u_j. Now, let there be a vertex u_k, $1 \leq k \leq n\text{-}m\text{-}1$, such that no vertex v_i, $1 \leq i \leq m$, has an arc from v_i to u_k. Since T is tournament, there is an arc from u_k to each of the m vertices v_i. Also, there is an arc from u_k to v and therefore $d^+(u_k) \geq m+1$. This contradicts the fact that v has maximum out degree with $d^+(v) = m$. Thus each u_j must have an arc joining it from some v_i and the proof is complete by using the directed path vv_iu_j.

Definition

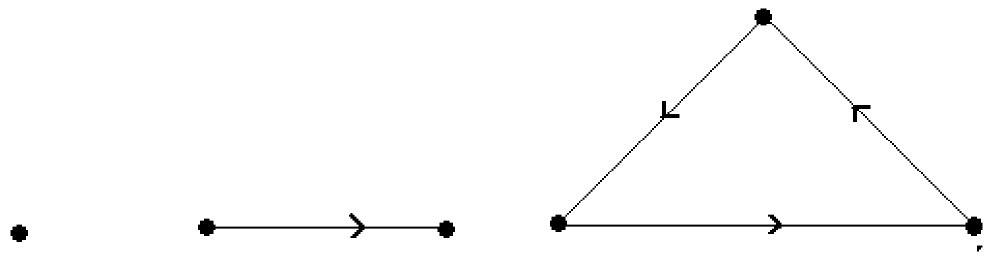
A directed Hamiltonian path of a digraph D is the directed path in D that includes every vertex of D exactly once.

THEOREM

Every tournament T has a directed Hamiltonian path.

Proof

Let T be a tournament with n vertices. The proof is by mathematical induction on n. When n = 1,2 or 3, result is trivially true.



Let $n \geq 4$. Assume that the result is true for all tournaments with fewer than n vertices. Let v be any vertex of T. Then T–v is a tournament with n-1 vertices and by induction hypothesis has a directed Hamiltonian path, say, P = $v_1 v_2 \dots v_{n-1}$.

In case there is an arc from v to v_i, then $P_1 = v v_1 v_2 \dots v_{n-1}$ is a directed Hamiltonian path in T. Similarly, if there is an arc from v_{n-1} to v, then $P_2 = v_1 v_2 \dots v_{n-1} v$ is a directed Hamiltonian path in T.

Now, assume there is no arc from v_{n-1} to v. Then there is at least one vertex w on the path P with the property that there is an arc from w to v and w is not v_{n-1}. Let v_i be the last vertex on P having this property, so that the next vertex v_{i+1} does not have this property. Then there is an arc from v_i to v and an arc from v to v_{i+1}.

Thus $Q = v_1 v_2 \dots v_i v_{i+1} v_{i+2} \dots v_{n-1}$ is a directed Hamiltonian path in T. Hence the proof.

Definition

A directed Hamiltonian cycle in a digraph D is a directed cycle which includes every vertex of D. If D contains such a cycle, then D is called Hamiltonian.

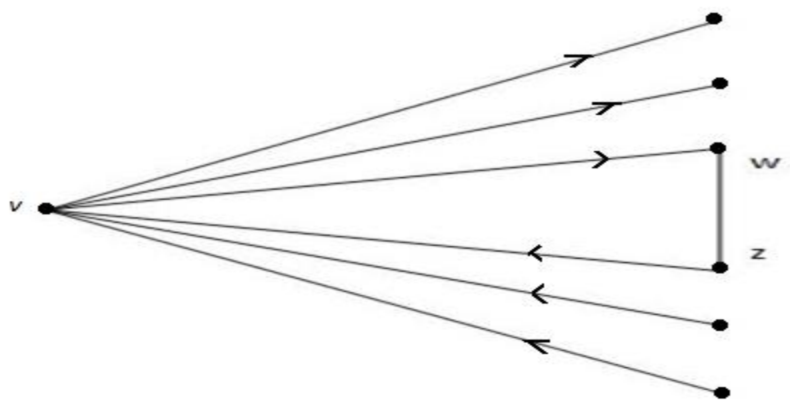
THEOREM:1

A strongly connected tournament T on n vertices contains cycles of length 3, 4, … n.

Proof

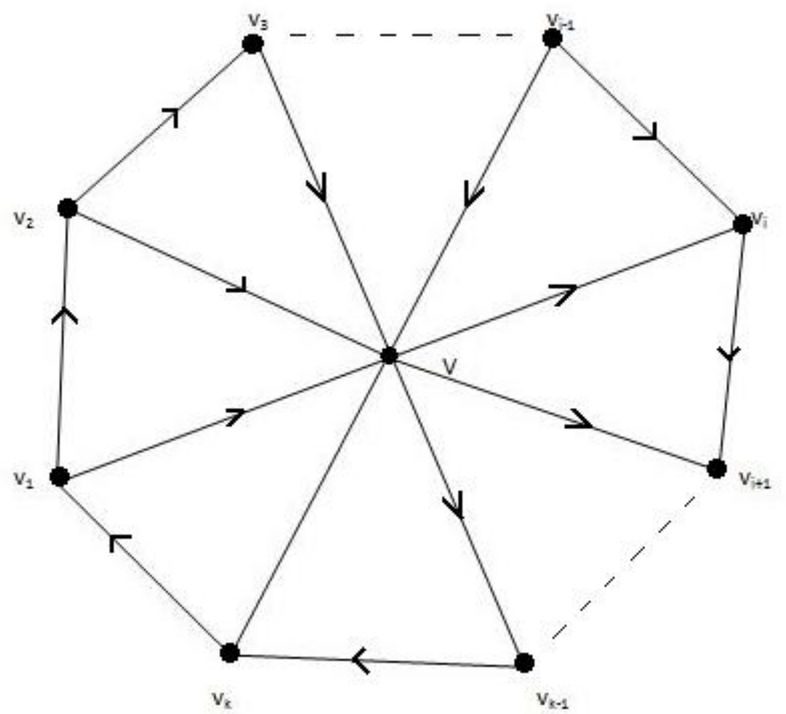
First we show that T contains a cycle of length three. Let v be any vertex of T. Let W denote the set of all vertices w of T for which there is an arc from v to w. Let Z denote the set of all vertices z of T for which there is an arc from z to v. We note that W∩Z = φ, since T is a tournament.

Since T is strongly connected, W and Z are both nonempty. For, if W is empty , then there is no arc going out of , then there is no arc going out of v, which is impossible because T is strongly connected and some argument can be used for Z. Again, because T is strongly connected, there is an arc in T going from w in W to some z in Z. This gives the directed cycle $vwzv$ of length 3.

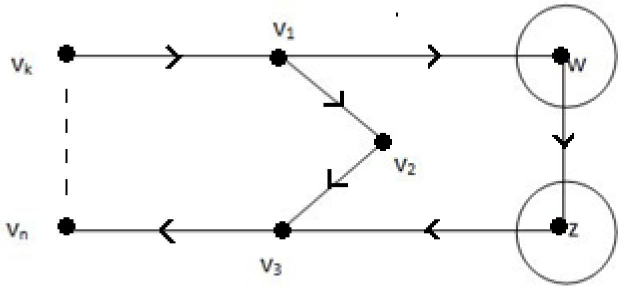


We now use induction to finish the proof. Assume T has a cycle C of length k, where $k < n$ and $k \geq 3$ and let this cycle be $v_1 v_2 \ldots v_k v_1$. We show that T has a cycle of length k+1.

Let there be a vertex v not on the cycle C, with the property that there is an arc from v to v_i and an arc from v_j to v for some v_i, v_j on C. Then there is a vertex v_i on C with an arc of length k+1.



If no vertex exists with the above property, then the set of vertices not contained in the cycle can be divided into two distinct sets W and Z, where W is the set of vertices w such that for each i, $1 \leq i \leq k$, there is an arc from v_i to w and Z is the set of vertices z such that for each i, $1 \leq i \leq k$, there is an arc from z to v_i. If W is empty then the vertices of C and the vertices of Z together make up all the vertices in T. But, by definition of Z, there is no arc from a vertex on C to a vertex in Z, a contradiction, because T is strongly connected. Thus W is nonempty. A similar argument shows that Z is nonempty. Again, since T is strongly connected, there is an arc from some w in W to some z in Z. Then C_1=$v_1 w z v_3 v_4 \dots v_k v_1$ is a cycle of length k+1. This complete the proof.



THEOREM

A tournament T is Hamiltonian if and only if it is strongly connected.

Proof

Let T have n vertices. If T is strongly connected, then by theorem-1, T has a cycle of length n. Such a cycle is Hamiltonian cycle, since it includes every vertex of T. Hence T is Hamiltonian.

Conversely, let T be Hamiltonian with Hamiltonian cycle, C=$v_1 v_2 \dots v_n v_1$. Then given any v_i, v_j with $i \geq j$, in the vertex set of T, $v_j v_{j+1} \dots v_i$ is a path P_1 from v_j to v_i while $v_i v_{i+1} \dots v_{n-1} v_n v_1 \dots v_{j-1} v_j$ is a path P_2 from v_i to v_j.Hence the proof.

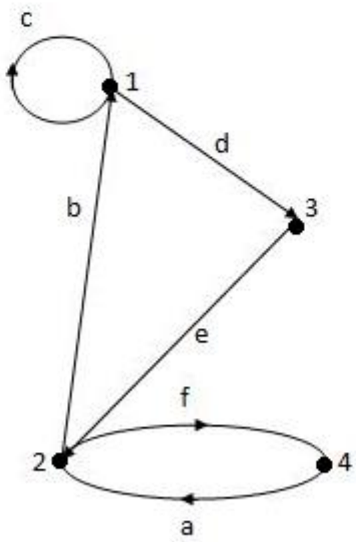
CHAPTER - 3

MORE ON DIGRAPHS

3.1 EULER DIGRAPHS

The notion of the Euler graph can be extended to digraphs also .In a digraph G a closed directed walk (i.e, a directed walk that starts and ends at the same vertex) which traverses every edge of G exactly once is called a directed **Euler line** .A digraph containing a directed Euler line is called an **Euler digraph**.

Example:



This is an Euler digraph, in which the walk $abcdef$ is an Euler line.

The digraph must be connected, with the possible exception isolated vertices, otherwise every edge can't be traversed in one walk. In fact, an Euler digraph must be strongly connected, although every strongly connected digraph need not be an Euler digraph.

THEOREM

A digraph D = (V , A) is Eulerian if and only if D is connected and for each of it's vertices V ,$d^-(v) = d^+(v)$

Proof:-

Necessary part

Let D be an Eulerian digraph. Therefore it contains an Eulerian walk, say W. In traversing W, every time a vertex v is encountered. We pass along an arc incident towards v and then an arc incident away from v. This is true for all the vertices of W, including the initial vertex of W,say v because we began W by traversing an arc incident away from v and ended W by traversing an arc incident towards v.

Sufficient part

Let for every vertex in D, $d^-(v)=d^+(v)$. For an arbitrary vertex v in D, we identify a walk, starting at v and traversing the arcs of D at most once each. This traversing is continued till it is impossible to traverse further.Since every vertex has the same number of arcs incident towards it as away from it. We can leave any vertex that we enter along the walk and the traversal then stops at v. Let the walk traversed so far be denoted by W .If W includes all the arcs of A, then the result follows. If, not we remove from D all the arcs directed and consider the remainder of A. By assumption, each vertex in the remaining digraph, say D, is such that the number of arcs directed towards it equals the number of arcs directed away from it. Further, W and D, have a vertex, say u in common. Since D is connected starting at u, we repeat the process of tracing a walk in D_1 . If this walk does not contain all the arcs ofD_1, the process is repeated until a closed walk that traverses each of the arcs of D exactly once is obtained.

Hence D is Eulerian.

THEOREM

A non trivial weak digraph is an isograph if and only if it is the union of arc-disjoint cycles.

Proof:-

If the weak digraph D is a union of arc-disjoint cycles each cycle contributes one to the in degree and one to the out degree of each vertex on it. Thus $d^{+}(v)=d^{-}(v)$,for all $v \in V$.

Conversely, let D be a non trivial weak isograph. Then each vertex has positive out degree and therefore D has a cycle, say Z. Removing the edges of Z from D, we get a digraph D_1, whose weak components are isographs. By using an induction argument, each such nontrivial weak component is a union of arc disjoint cycles. These cycles together with Z provide a decomposition of the arc set of D in to cycles.

COROLLARY

Every weak isograph is strong.

Proof:-

If u and v are any two vertices of the weak isograph D, there is a semi path P joining u and v and each arc of this lies on some cycle of D . The union of these cycles provides a closed walk containing u and v. This u and v are mutually reachable.

3.2 MATRICES A,B AND C OF DIGRAPHS

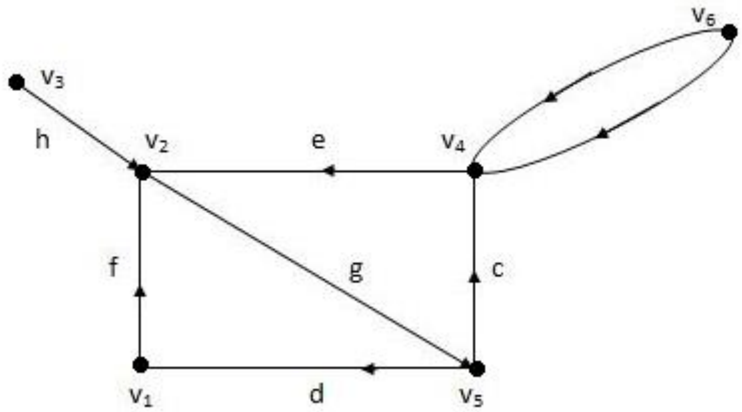
The matrices associated with a digraph are almost similar to those discussed for an undirected graph, with the difference that in matrices of diagraphs consist of 1,0,-1 instead of only 0 and 1 for undirected graphs . The numbers 1,0,-1 are real numbers and their addition and multiplication are interpreted as in ordinary arithmetic, not modulo 2 arithmetic as in undirected graphs. Thus the vectors and vector spaces associated with a digraph and it's sub digraphs are over the field of all real numbers, but not modulo 2.

INCIDENCE MATRIX

The incidence matrix of a digraph with n vertices, e edges and no self loops is an $n \times m$ matrix A, A=[a_{ij}], whose rows correspond to vertices and columns correspond to edges such that

$$a_{ij}=\begin{cases} 1\ , if\ j^{th}\ edge\ is\ incident\ out\ of\ i^{th}\ vertex \\ -1, if\ j^{th}\ edge\ is\ incident\ into\ i^{th}\ vertex \\ 0, if\ j^{th}\ edge\ is\ not\ incident\ on\ i^{th}\ vertex \end{cases}$$

For example, consider the digraph



The incidence matrix is given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Since the sum of each column is zero, the rank of the incidence matrix of a digraph of n vertices is less than n.

THEOREM

If A(G) is the incidence matrix of a connected digraph of n vertices, the rank of A(G)= n – 1

Proof:-

Deleting any one row from A we getA_f,the n – 1 by reduced incidence matrix . The vertex corresponding to the deleted row is called the reference vertex .

If A is incidence matrix of an undirected graph, the determinant was defined in modulo 2 arithmetic and therefore, could have no other value. In the case of diagraphs, the incidence matrix A is in the real field, and on first sight it would appear that the determinants of its square submatrices could acquire any integral value.

UNIMODULAR MATRIX

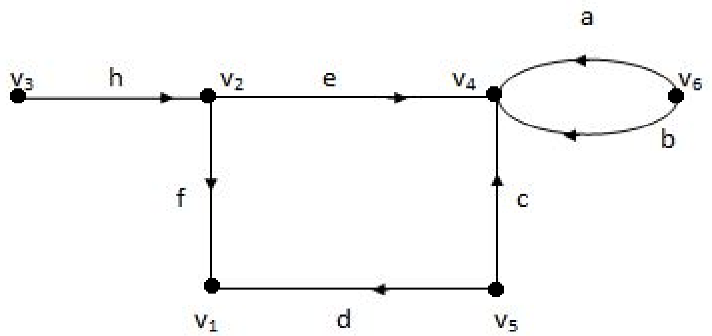
A matrix is said to be unimodular if the determinant of its every square matrix is -1,0 or 1 .

CYCLE MATRIX OF A DIGRAPH

Let G be a graph with e edges and q cycles (directed cycles or semi cycles). An arbitrary orientation (clockwise or counter clockwise) is assigned to each of the cycles q . Then a cycle matrix B= $[b_{ij}]$ of the digraph G is a q×e matrix defined as

$$b_{ij}=\begin{cases} 1\ , if\ the\ i^{th}\ cycle\ includes\ the\ j^{th}\ edge\ and\ the\ orientation \\ \quad of\ the\ edge\ and\ cycle\ coincide \\ -1\ , if\ the\ i^{th}\ cycle\ includes\ the\ j^{th}\ edge\ and\ the\ orientation \\ \quad of\ the\ two\ are\ opposite \\ 0\ , if\ the\ i^{th}\ cycle\ does\ not\ include\ the\ j^{th}\ edge \end{cases}$$

Eg:-Consider the digraph D in fig



One cycle of matrix D is

$$A = \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The cycle in the first row is assigned clockwise orientation, in the second row counter clockwise, in the third row counter clockwise and in the fourth clockwise. Changing the orientation of any cycle will simply change the sign of every non zero entry in the corresponding row. Also we observe that if the row is subtracted from second, the third is obtained. Thus the rows are not all linearly independent.

SIGN OF A SPANNING TREE

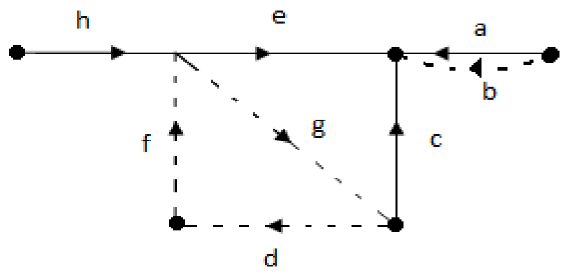
For a digraph, the determinant of the non singular submatrix of A corresponding to a spanning tree T has a value either 1 or -1 .This is referred to as sign of T.

The sign of a spanned tree is defined only for a particular ordering of vertices and edges in A, because interchanging two rows and columns in a matrix changes the sign of its determinant. Thus the sign of a spanning tree is relative. Once the sign of one spanning tree is arbitrarily chosen, the sign of every other spanning tree is determined as positive or negative with respect to this spanning tree.

FUNDAMENTAL CYCLE OF MATRIX

The M fundamental cycles each formed by a chord with respect to some specified spanning tree, define a fundamental cycle matrix B, for a digraph. The orientation assigned to each of the fundamental cycles is chosen to coincide with that of the chord. Therefore, B_fa μ×m matrix can be expressed exactly in the same form as in the case of an undirected graph.

$B_f = [I_\mu : B_t]$, where I_μ is the identity matrix of order M and the columns of B_t corresponds to the edges in a spanning tree.



$$\begin{array}{cccccccc} b & d & g & a & c & e & f & h \end{array}$$

Here $B_f = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} = [I_\mu : B_t]$

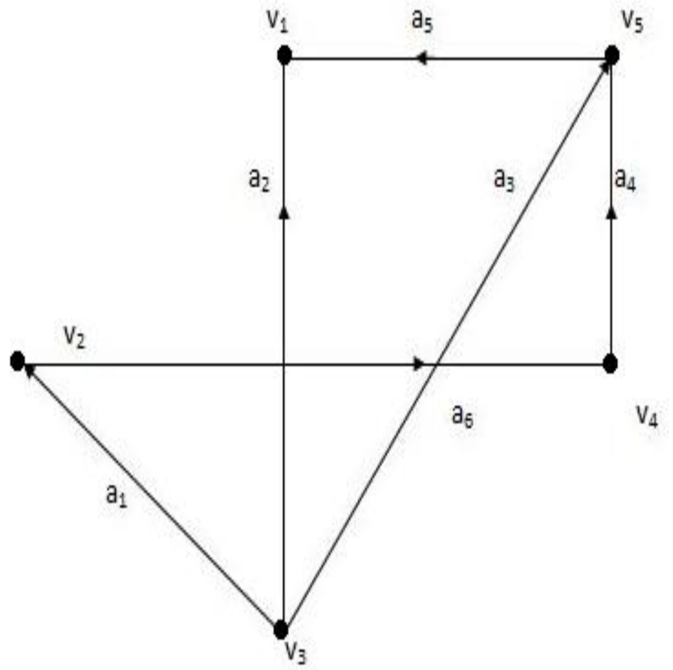
CUT-SET MATRIX

Let D= (V,A) be a connected digraph with q cut sets. The cut-set matrix C=[c_{ij}] of D is a q×m matrix in which the rows corresponds to the cut-sets of D and the columns to the edges of D. Each cut-sets given an arbitrary orientation. Let R_i be the i^{th} cut –set of D and R_i partition V into non empty vertex sets $v_i'$and v_i''. The orientation can be defined to be either from $v_i'$to v_i'' or from v_i'' tov_i' . Suppose the orientation is chosen to be from. Then the orientation of an edge of a_j cut-set is$R_i$said to be the same as that of $R_i$if $a_j$is of the form$v_a v_b$, where $v_a \in v_b'$and $v_b \in v_i''$and opposite, otherwise . Then,

$$c_{ij} = \begin{cases} 1, \text{if an edge } a_j \text{ of cut} - \text{set } R_i \text{ has the same orientation as } R_i \\ -1, \text{if edge } a_j \text{ has the opposite orientation to } R_i \\ 0, \text{ otherwise} \end{cases}$$

SEMI-PATH MATRIX

The semi-path matrix P(u,v) = [p_{ij}] of a digraph D=(V ,A) where u, v ∈V , is the matrix with each row representing a distinct semi-path from u to v and the columns representing the arcs of D, in which p_{ij}=1 if the i^{th} semi-path contains the j^{th} edge ,p_{ij} = -1 if the semi-path contains the converse of the j^{th} edge and p_{ij} =0 otherwise .



The matrix P (V_3, V_5) for the digraph of above figure

$$a_1\ a_2\ ra_3\ \ a_4\ \ a_5\ a_6$$

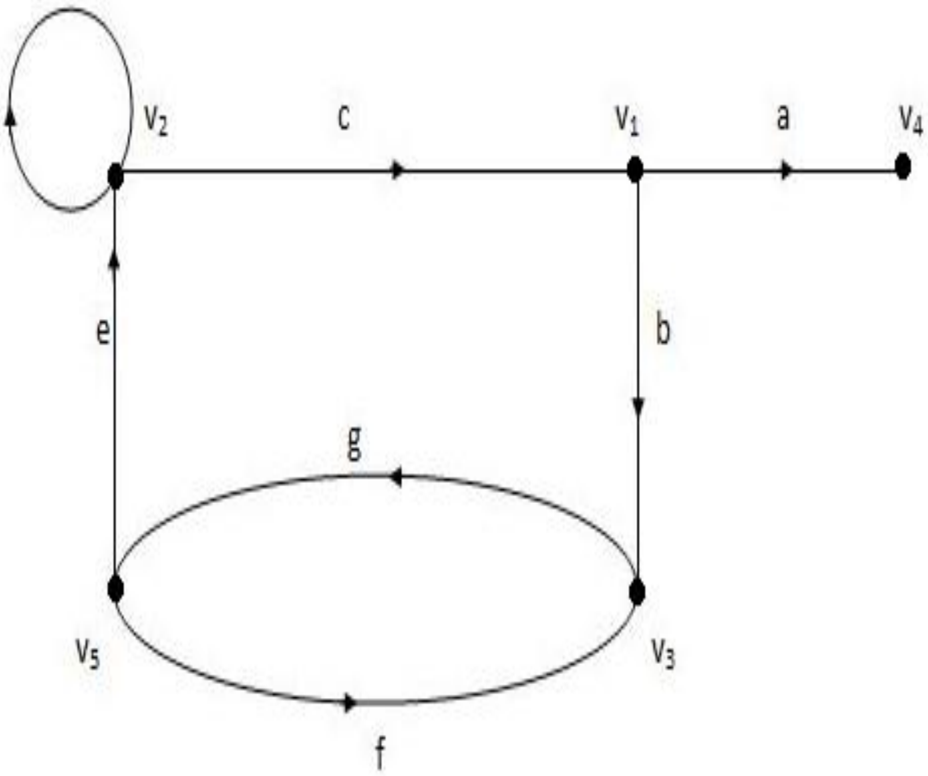
$$P\ (V_3, V_5) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

ADJACENCY MATRIX OF A DIGRAPH

Let G be a digraph with n vertices and with no parallel edges. The adjacency matrix X = $[x_{ij}]$ of the digraph G is an n×n (0,-1) matrix defined by,

$$x_{ij} = \begin{cases} 1 \text{ , if there is an edge from i}^{\text{th}} \text{ vertex to j}^{\text{th}} \text{ vertex} \\ 0, \text{otherwise} \end{cases}$$

Eg:- Consider the figure



The adjacency matrix of D is

	V1	V2	V3	V4	V5
V1	0	0	1	1	0
V2	1	1	0	0	0
V3	0	0	0	0	1
V4	0	0	0	0	0
V5	0	1	1	0	0

CHAPTER – 4

APPLICATIONS

The concept of digraph (or directed graphs) is one of the richest theories in graph theory, mainly because of their applications to physical problems. For example, the street map of a city with one-way streets, Flow networks with values in the pipes, and electrical networks are represented by directed graphs. Directed graphs are employed in abstract representation of computer programs, where the vertices stand for the programs instructions and the edges specify the execution sequence. The directed graph is an invaluable tool in the study of sequential machines. Directed graphs in the form of signal flow graphs are used for system analysis in control theory.

Digraphs under the name sociograms have been used to represent relationships among individuals in a society (or group). Members are represented by vertices and the relationship by anthropologists and are classified according to their kinship structures.

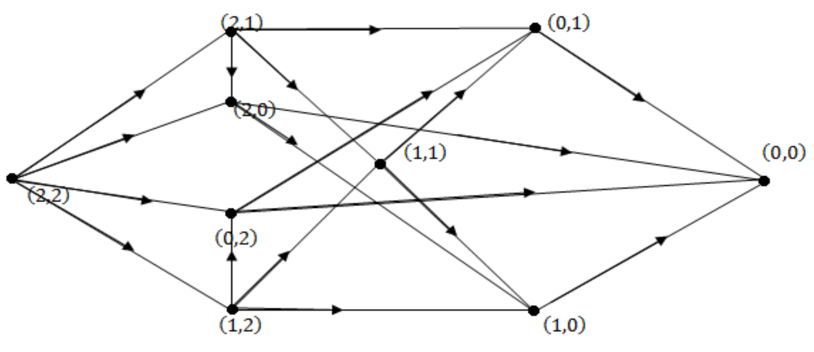
The concept of directed graphs have become a legitimate and very useful area of operational research (OR). As OR is being applied to more and more problems of society, it is apparent that digraphs models and algorithms have the potential to be of great use in the social sciences. The three important classes of problems in combinational operations research transportation problems, activity networks and game theory can be expressed and solved elegantly as graph theory problems involving connected and weighted digraphs.

IN GAME THEORY

The theory of game has become an important field of mathematical research since the publication of the first book on the subject by **John von**

Neumann and Oskar Morgenstern in 1944.Game theory is applied to problems in engineering, economics and war science to find the optimal way of performing certain tasks in a competitive environment. The general idea of game theory is the same as the one we associate with parlor game such as Chess bridge and checkers

Simplified Nim: 2 piles of sticks are given and players A and B take turns each taking any number of sticks from any one pile. The player who takes the last stick win and since the finite quantity of sticks will eventually be exhausted it is obvious that the game allows no draw. As a further simplification, let as start with two piles containing two sticks each. The complete game is described by the digraph. Each state of the game is described by an ordered pair of labels (x, V), indicating the number of sticks in the first and the second pile, respectively.



1. The digraph has a unique vertex with a zero in-degree. This vertex represents the starting position in the game and is therefore called the starting vertex. Vertex (2, 2)

2. There are one or more vertices with zero out-degree. 1hese correspond to the closing positions in the game, and are called the closing vertices Vertex (0, 0)

3.A game digraph is a connected, acyclic digraph. A directed circuit would imply that the game could go on indefinitely.

Each directed path from the starting vertex to a closing vertex represents one complete play of the game. This path consists of edges representing the moves of the two players alternately.

Let us call a position "won" if the player who brought the game to this position Can force a victory. Conversely, a position is dubbed "lost if the player who brought the game to this position can be forced to lose. the closing vertex marked as won, because the player who brought the game to this position is the winner. Having marked this vertex as won, let us use the following procedure to mark the remaining vertices as won or lost. Mark an unmarked vertex won if all its successors are marked lost, and mark an unmarked vertex lost if at least one of its successors is marked won. This results in vertices (0, 0), (1, 1), and (2, 2) being marked as won and the remaining as lost. And thus the player who makes the second move has the Winning strategy, Since he can force his opponent to move to the vertices marked as lost.

SIGNAL-FLOW GRAPHS

Most problems in analysis of a linear system are eventually reduced to solving a set of simultaneous, linear algebraic equations. This problem usually solved by matrix methods, can also be solved via graph theory. The graph-theoretic approach is often faster, and, more importantly, it displays cause-effect relationships between the variables-something totally obscured the matrix approach. This graph-theoretic analysis of a linear system consists of two parts: (1) constructing a labeled, weighted digraph called the graph, and (2) solving tor the required dependent variable from the signal-flow graph.

In a signal-flow graph each vertex represents a variable and is labeled so. A directed edge from x_i to x_j Implies that variable x_i depends on variablex_j.

The coefficients in the equations are assigned as the weights of the edges Such that the variable x_k is equal to the sum of all products $W_{ik}x_i$ where W_{ik} is the weight of the edge coming into x_k from x_i. As an example, let us construct a signal-flow graph for the system given by the set of three equations,

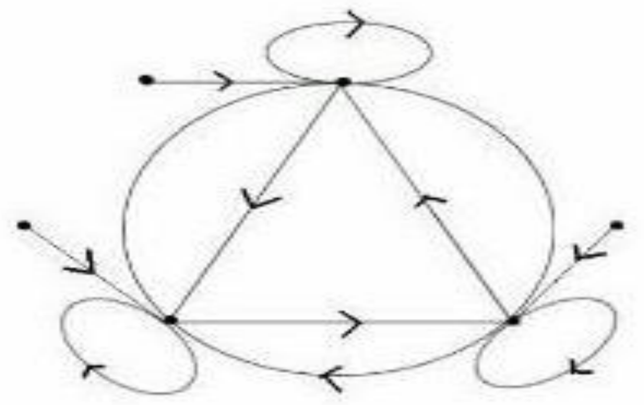
{C11X1+C12X2+C13X3=Y1

C21X1+C22X2+C23X3=
Y2
C31X1+C32X2+C33X3=
Y3} –(a) which can be
rewritten as
{(C11+1)X1+C12X2+C13X3-Y1=X1

C21X1+(C22+1)X2+C23X3-Y2=X2

C31X1+C32X2+(C33+1)X3-Y3=X3} –(b)

The graph representing equation- (a)

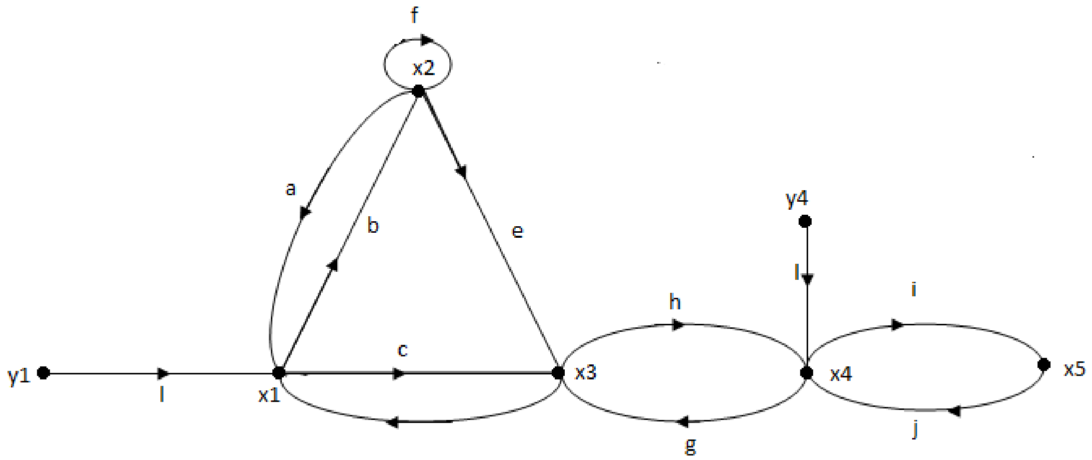


<u>THEOREM</u>

The weight matrix W=[w_{ij}] of the signal-flow graph corresponding to equation (**c)** is given by

$$\mathbf{W} = \begin{bmatrix} c+1 & -1 \\ 0 & 0 \end{bmatrix}$$ **equation. [d]**

where 1 is the identity matrix of the same order as C, and the superscript T denotes the transposed matrix. Although signal-flow graphs can always be constructed from a set of equation in many physical problems, particularly in electrical systems signal-flow graphs are drawn directly without first writing the equations. Usually, a signal-flow graph can be drawn as easily as the equations are formulated. Also, writing equation from a signal-flow graph is a Simple matter, because each vertex $X_k$represents one equation or the system in which X_k is equal to the sum of the products of weights of all incoming edges and the labels of the initial vertices of these edges



$$x_1 = y_1 + ax_2 + dx_3$$

$$x_2 = bx_1 + fx_2$$

$$x_3 = cx_1 + ex_2 + gx_4$$

$$x_4 = hx_3 + jx_5 + y_4$$

$$x_5 = ix_4$$

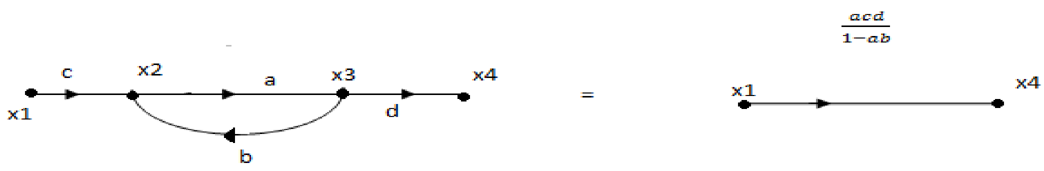
These can be rewritten in the same form as equ(c)

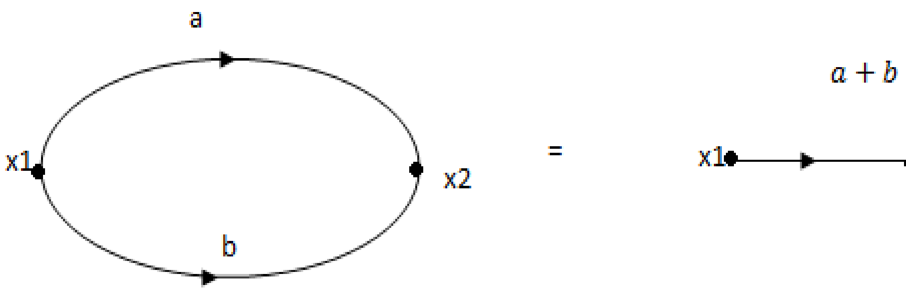
$$C = \begin{bmatrix} 1 & -a & -d & 0 & 0 \\ -1 & 1-f & 0 & 0 & 0 \\ -c & -e & 1 & -g & 0 \\ 0 & 0 & -h & 1 & -j \\ 0 & 0 & 0 & -i & 1 \end{bmatrix}$$

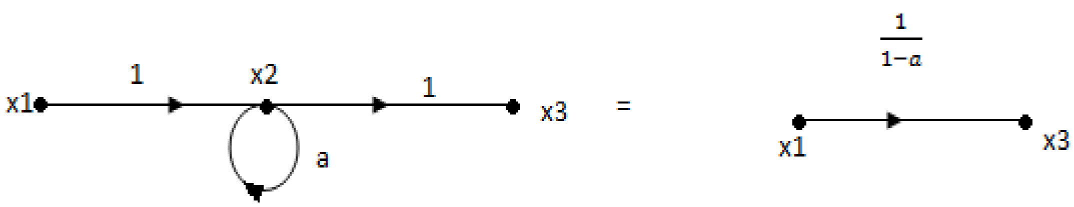
$$X = \begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \\ x5 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y1 \\ 0 \\ 0 \\ y4 \\ 0 \end{bmatrix}$$

<u>REDUCTION OF SIGNAL FLOW GRAPH</u>

The signal –flow graph method of analysis Is most useful when we want to solve for only one Unknown variable, say R J, as a function of one Independent variable say Y K we solve by eliminating All other vertices one by one taking care that this Elimination process does not alter the net product Of the edge weights of directed paths from YK to R j this graph reduction corresponds exactly to The algebras method of eliminating all other variable by systematic substitution.







Although our ability to reduce the digraph by simple inspection adds much to the power and flexibility of signal-flow graphs, it is often better to use a more methodical technique that does not depend on visual inspection. And such a method is provided by Mason's gain formula.

CONCLUSION

Most of the important and fundamental features of directed graphs were investigated in this project. We saw that there are two different aspects of digraphs: one in which their properties are similar to those of undirected graphs and second aspect have properties altogether different from those of undirected graphs. The close relationship between binary relations and digraphs was explored. Also studies on Euler digraphs, tournaments and matrix representation of digraphs are included applications of digraphs are virtually unlimited.

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2. Narsingh Rao, *Graph Theory With Applications To Engineering And Computer Science*, Prentice Hall of India Pvt. Ltd., 1974