APPLICATIONS OF DIFFERENTIAL EQUATIONS

PROJECT SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR

THE BACHELOR OF SCIENCE DEGREE IN MATHEMATICS 2017-2020

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CERTIFICATE

This is to certify that the project report entitled "APPLICATIONS OF DIFFERENTIAL EQUATIONS" is a bonafide record of studies undertaken by DIVYA S (Reg No.170021032407), GOPIKA LIGHTO (Reg No. 170021032411)**,** MEERA P (Reg No. 170021032420) in partial fulfilment of the requirements for the award of B.Sc. Degree in Mathematics at Department of Mathematics, St. Paul's College, Kalamassery, during the academic year 2017-2020.

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DECLARATION

We, DIVYA S (Reg no. 170021032407), GOPIKA LIGHTO (Reg no. 170021032411), MEERA P (Regno.170021032420) hereby declare that this project entitled "APPLICATIONS OF DIFFERENTIAL EQUATIONS" submitted to Department of Mathematics of St. Paul's college, Kalamassery in partial requirement for the award of B.Sc Degree in Mathematics, is a work done by us under the guidance and supervision of Dr. SAVITHA K.S , Department of Mathematics, St. Paul's college, Kalamassery during the academic year 2017-2020.

We also declare that this project has not been previously presented for the award of any other degree, diploma, fellowship, etc.

KALAMASSERY

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ACKNOWLEDGEMENT

For any accomplishment or achievement the prime requisite is the blessing of the Almighty and it's the same that made this world possible. We bow to the lord with a grateful heart and prayerful mind.

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INTRODUCTION

In mathematics, a differential equation is an equation that relates one or more functions and their derivatives. In applications, the functions generally represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two. Such relations are common, therefore differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.

Mainly the study of differential equations consists of the study of their solutions (the set of functions that satisfy each equation), and of the properties of their solutions. Only the simplest differential equations are solvable by explicit formulas; however, many properties of solutions of a given differential equation may be determined without computing them exactly.

Often when a closed-form expression for the solutions is not available, solutions may be approximated numerically using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

The importance of a differential equation as a technique for determining a function is that if we know the function and possibly some of its derivatives at a particular point, then this information, together with the differential equation can be used to determine the function over its entire domain. Differential equations have wide applications in various engineering and science disciplines.

In general, modelling variations of a physical quantity, such as temperature, pressure , displacement, velocity, stress, strain, or concentration of a pollutant, with the change of time t or location, such as the coordinates (x,y,z), or both would require differential equations. Similarly, studying the variation of a physical quantity on other physical quantities would lead to differential equations. For example, the change of strain on stress for some visco-elastic materials follows a differential equation.

In this project, a few examples are presented to illustrate how practical problems are modelled mathematically and how differential equations arise in them.

CHAPTER ONE

PRELIMINARIES AND DEFINITIONS

- A **Differential equation** is an algebraic relation involving derivatives of one or more unknown functions with respect to one or more independent variables, and possibly either the unknown functions themselves or their independent variables, that holds at every point where all of the functions appearing in it are defined.

- A differential equation is called an **ordinary differential equation (ODE)** if it involves derivatives with respect to only one independent variable. Otherwise, it is called a partial differential equation (PDE).

- A **solution** (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation.

- The **order** of a differential equation is the order of the highest derivative that appears in it. An nth-order differential equation is one whose order is n.

- The **degree** of differential equation is represented by the power of the highest order derivative in the given differential equation.

- A **first-order differential equation** is defined by an equation

$\frac{dy}{dx} = f(x, y)$ of two variables x and y with its function $f(x, y)$ defined on a region in the xy-plane. It has only the first derivative $\frac{dy}{dx}$, so that the equation is of the first order and not higher-order derivatives.

- A differential equation is said to be **linear** if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus the standard form of a linear equation of the first order, is

$$\frac{dy}{dx} + Py = Q$$

where P, Q are the functions of x.

- A first order differential equation is said to be **homogeneous** if it may be written $f(x, y)dy = g(x, y)dx$ where f and g are homogeneous functions of the same degree of x and y.
- First order ordinary differential equations can be solved by method of separation of variables followed by integration.

- The **second order homogeneous linear differential equation** with constant coefficients is $ay'' + by' + c = 0$

where y is an unknown function of the variable x, and a, b, and c are constants. If $a = 0$ this becomes a first order linear equation, which in this case is separable, and so we already know how to solve.

So we will consider the case $a \neq 0$

Consider the equation,

$$ay'' + by' + c = 0$$

Step 1-Form the quadratic equation $ar^2 + br + c = 0$

It is called the Associated Polynomial Equation or Auxiliary Equation (A.E)

Step 2-Find the roots of the A.E.

Several cases arise.

Case 1: The A.E. has distinct real roots a and b , then the solution is

$$y = A_1 e^{ax} + A_2 e^{bx}$$

Case 2 : The A.E has repeated roots(say a), then the solution is

$$y = (A_1 + A_2 x) e^{ax}$$

Case 3 : The A.E. has complex roots $a \pm ib$, then the solution is

$$y = e^{ax}(A_1 \cos bx + A_2 \sin bx)$$

- **Simple harmonic motion:** Is an oscillatory motion under a retarding force which is proportional to the amount of displacement from an equilibrium position
- The **restoring force** is a force which acts to bring a body to its equilibrium position. The restoring force is a function only of position of the mass or particle, and it is always directed back toward the equilibrium position of the system. The restoring force is often referred to in simple harmonic motion.

- The restraining of vibratory motion, such as mechanical oscillations, noise, and alternating electric currents, by dissipation of energy is known as **damping force**.

Overdamped: The system returns to equilibrium without oscillating

Critically damped: The system returns to equilibrium as quickly as possible without oscillating.

Underdamped: The system oscillates with the amplitude gradually decreasing to zero.

In classical mechanics, a **harmonic oscillator** is a system that, when displaced from its equilibrium position, experiences a restoring force F proportional to the displacement x. If F is the only force acting on the system, the system is called a **simple harmonic oscillator**, and it undergoes simple harmonic motion: sinusoidal oscillations about the equilibrium point, with constant amplitude and a constant frequency (which does not depend on the amplitude).

If a frictional force (damping) proportional to the velocity is also present, the harmonic oscillator is described as a **damped**

oscillator. Depending on the friction coefficient, the system can be underdamped or critically damped or overdamped.

Mechanical examples include pendulums (with small angles of displacement), masses connected to springs, and acoustical systems. Other analogous systems include electrical harmonic oscillators such as RLC circuits. The harmonic oscillator model is very important in physics, because any mass subject to a force in stable equilibrium acts as a harmonic oscillator for small vibrations.

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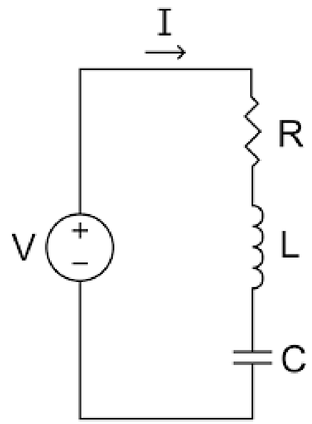
- An **LCR circuit** is an electrical circuit consisting of a resistor, an inductor, and a capacitor, connected in series or in parallel.



- **Kirchhoff's I law:**
  The law states that current flowing into a node (or a junction) must be equal to current flowing out of it. This is a consequence of charge conservation.

- **Kirchhoff's II law**: The law states that the sum of all voltages around any closed loop in a circuit must equal zero.
- In LCR circuit, the voltage in the capacitor eventually causes the current flow to stop and then flow in the opposite direction. The result is an oscillation, or resonance.

Damping means waning or reduction of stored energy or current/ pulse in a circuit. At Resonance, LC combination gives rise to oscillations, which continue indefinitely in ideal LC circuit. However LCR circuit has energy consumer components R, which converts energy to heat, reducing electrical energy.

- As the charge contained in the LCR circuit oscillates back and forth through the resistance, electromagnetic energy is dissipated as thermal energy, therefore damping (decreasing the amplitude of) the oscillations. The capacitor is initially charged to Q_0.
- The circuit is said to be overdamped because two superimposed exponentials are both driving the current to zero. A circuit will be overdamped if the resistance is high relative to the resonant frequency.
- **Partial Differential Equations**
  If the dependent variable u is a function of more than one independent variable, say $x_1, x_2, x_3, \ldots . x_m$, an equation involving the variables $x_1, x_2, x_3, \ldots . x_m, u$ and various partial derivatives of u with respect to $x_1, x_2, x_3, \ldots . x_m$, is called a Partial Differential Equation (PDE).

- The general **second order partial differential equations** in two variables is of the form

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0$$

- **Wave Equation:** A differential equation expressing the properties of motion in waves

CHAPTER TWO

Applications of First-Order Differential Equations

In this we will see some applications of first order differential equations.

Motion of a Particle in a Resisting Medium

Newton's Second Law and D'Alembert's Principle Newton's Second Law: The product of the mass of an object and its acceleration is equal to the sum of forces applied on the object, i.e., $ma = \sum F$. D'Alembert's Principle: Rewrite Newton's Second Law as $\sum F - ma = 0$. Treat $-ma$ as a force, known as the inertia force. An object is in (dynamic) equilibrium under the action of all the forces applied, including the inertia force. This is known as D'Alembert's Principle, which transforms a problem in dynamics into a problem of static equilibrium.

Impulse-Momentum Principle For a system of particles- The change in momentum of the system is equal to the total impulse on the system,

i.e., (Momentum at timet_2)−(Momentum at time t_1)=(Impulse during $t_2 - t_1$).

The momentum of a mass m moving at velocity $v$is equal to mv. The impulse of a force Fduring time interval $t$is equal to$F\Delta t$.

Consider the motion of a particle moving in a resisting medium, such as air or water. The medium exerts a resisting force R on the particle. In many applications, the resisting force R is proportional to v^n, where v is the velocity of the particle and $n > 0$, and is opposite to the direction of the velocity. Hence, the resisting force can be expressed as $R = \beta v^n$, where β is a constant.

For particles moving in an unbounded viscous medium at low speed, the resisting force is $R=\beta v$, i.e., $n = 1$. In the following, the case with $R = \beta v$ will be studied for motion in the vertical direction and specific initial conditions.

Let us determine the differential equation that governs the motion of the particle.

Case I: Upward Motion

Consider an object being launched vertically at time $t=0$ from $x=0$ with initial velocity v_0 as shown in Figure 1. The displacement x, the velocity $v = \dot{x}$, and the acceleration $a = \dot{v} = \ddot{x}$; $\left(\dot{v} = \frac{dv}{dt}, \ddot{x} = \frac{d^2x}{dt^2}\right)$ are taken as positive in the upward direction. The particle is subjected to two forces: the downward gravity mg and the resisting force from the medium $R = \beta v$, which is opposite to the direction of the velocity and hence is downward.

From Newton's Second Law, the equation of motion is

$$\uparrow ma = \sum F\,;$$

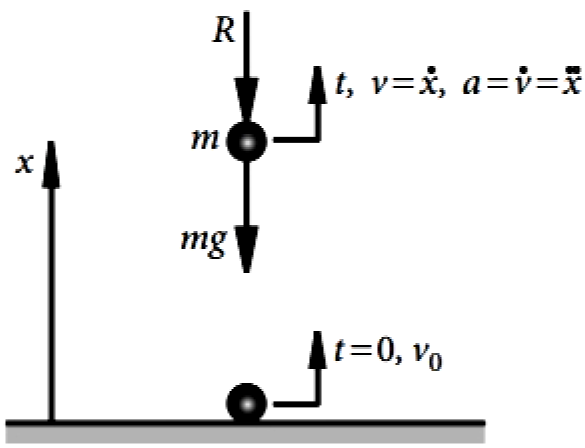


Figure 1 Upward motion of a particle in a resisting medium.

$$m\frac{dv}{dt} = -R - mg \quad ; \qquad R = \beta v \, , m = \frac{w}{g},$$

$\frac{dv}{dt} = -gt(\alpha v + 1), \alpha = \frac{\beta}{v} > 0$,Variable separable

$$\int \frac{dv}{(\alpha v + 1)} = -\int g dt + C; \; \therefore \frac{1}{\alpha}\ln(\alpha v + 1) = -gt + C. \qquad (1)$$

Constant C is determined from the initial condition $t = 0, v = v_0$;

$$\frac{1}{\alpha}\ln(\alpha v_0 + 1) = 0 + C$$

Substituting into equation (1) yields

$$\frac{1}{\alpha}\ln(\alpha v + 1) = -gt + \frac{1}{\alpha}\ln(\alpha v_0 + 1)$$

$\frac{1}{\alpha}\ln\left(\frac{\alpha v + 1}{\alpha v_0 + 1}\right) = -gt$ i.e. $\frac{(\alpha v + 1)}{(\alpha v_0 + 1)} = e^{-\alpha g t}$

Solving for v leads to $\alpha v = e^{-\alpha g t}(\alpha v_0 + 1) - 1$ (2)

When the object reaches maximum height at time t=t_{max}, v = 0 and

$$\alpha . 0 = e^{-\alpha g t_{max}}(\alpha v_0 + 1) - 1 \quad \rightarrow t_{max} = \frac{1}{\alpha g}\ln(\alpha v_0 + 1)$$

To determine the displacement x(t) .note that $v = \frac{dx}{dt}$ and use equation (2)

$\frac{dx}{dt} = \frac{1}{\alpha}(\alpha v_0 + 1)e^{-\alpha g t}$- $\frac{1}{\alpha}$

Integrating with respect to x gives

$x = \frac{1}{\alpha}(\alpha v_0 + 1)\int e^{-\alpha g t}\, dt - \frac{t}{\alpha} + \mathrm{D} = \frac{1}{\alpha^2 g}(\alpha v_0 + 1)e^{-\alpha g t} - \frac{t}{\alpha} + D$

Constant D is determined from initial conditions $t = 0$, $x = 0$:

$$0 = -\frac{1}{\alpha^2 g}(\alpha v_0 + 1)e^0 - \frac{0}{\alpha} + D \therefore D = \frac{1}{\alpha^2 g}(\alpha v_0 + 1).$$

Hence $x = -\frac{1}{\alpha^2 g}(\alpha v_0 + 1)(e^{-\alpha g t} - 1) - \frac{t}{\alpha}$

At time t=t_{max},the object reaches the maximum height given by

$x = x_{max} = x(t)|t = t_{max}$

$$= -\frac{1}{\alpha^2 g}(\alpha v_0 + 1)\left(e^{-\ln(\alpha v_0 + 1)} - 1\right) - \frac{1}{\alpha}.\frac{1}{\alpha g}\ln(\alpha v_0 + 1)$$

$= \frac{1}{\alpha^2 g}[\alpha v_0 - \ln(\alpha v_0 + 1)].$

Case II Downward Motion

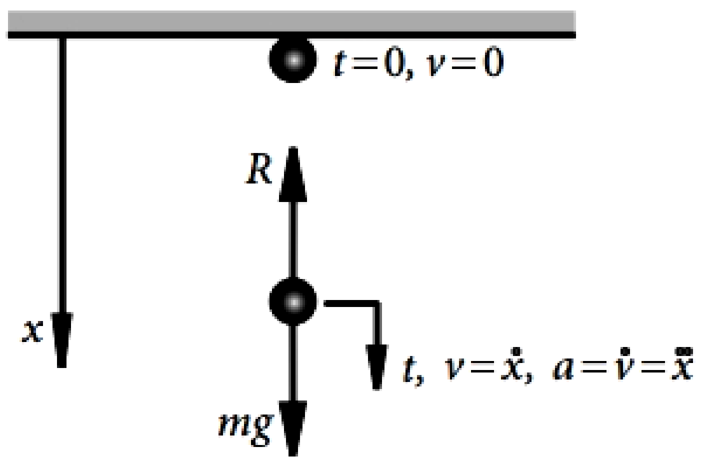


Figure 2 Downward motion of a particle in a resisting medium.

Consider an object being released and dropped at time $t = 0$ from $x = 0$ with $v = 0$ as shown in Figure .2. In this case, it is more convenient to take$x$, v, and$a$ as positiveinthedownwarddirection. Newton'sSecondLawrequires

$$\downarrow ma = \sum F; \qquad m\frac{dv}{dt} = mg - R, \quad R = \beta v, \qquad m = \frac{w}{g},$$

$\therefore\ \frac{dv}{dt} = g - \alpha g v\ , \quad \alpha = \frac{\beta}{w} > 0$ (variable separable)

The equation can be solved as

$$\int \frac{dv}{1 - \alpha v} = \int g dt\ + C$$

$$-\frac{1}{\alpha} ln|1 - \alpha v| = gt + C,$$

Where the constant C is determined from initial condition $t = 0, v = 0$

$$-\frac{1}{\alpha} ln1 = 0 + C\ ; \ \therefore C = 0$$

$$-\frac{1}{\alpha} ln|1 - \alpha v| = gt \quad \therefore v = \frac{1}{\alpha}(1 - e^{-\alpha g t}) \qquad (3)$$

When time t approaches infinity ,the velocity approaches a constant , the so called terminal velocity ,

$$v = v_{terminal} = \lim_{t \to \infty} v = \frac{1}{v}$$

The change of velocity with time is shown in figure 3

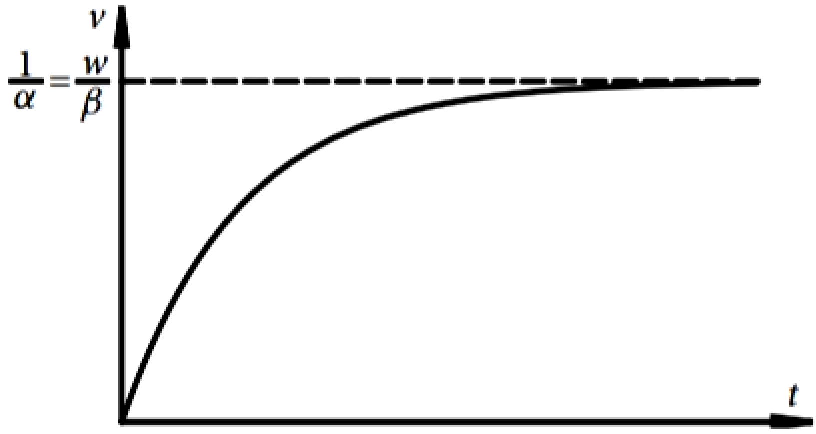
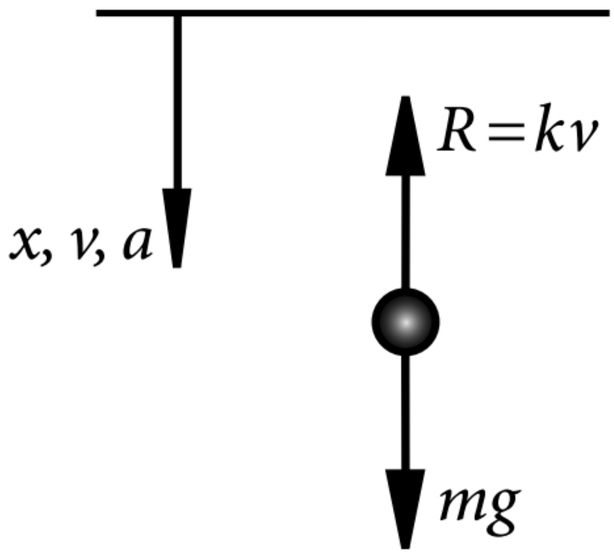


Figure 3 Velocity of a particle moving downward in a resisting medium.

Example – Object Falling in Air

An object of m falls against air resistance which is proportional to speed (i.e. $R = \beta v$) under gravity g.

If v_0 and v_E are the initial and final (terminal) speeds, v is the speed at time t,show that $\frac{v-v_E}{v_0-v_E} = e^{-kt}, \quad k = \frac{\beta}{m}$.



Answer: The object is subjected to two forces as shown; the downward gravity mg and the upward air resistance βv. Newton's Second Law requires

$$\downarrow ma = \sum F; \qquad m\frac{dv}{dt} = mg - \beta v\ ,$$

$$i.e\frac{dv}{dt} = g - kv\ ,\ \ k = \frac{\beta}{m}$$

Noting that $g - kv > 0$, the equation is variable separable and the solution is

$$\int \frac{dv}{g - kv} = \int dt + \ C\ ; \quad -\frac{1}{k}\ln(g - kv) = t + C\ ;$$

$$\therefore v = \frac{g}{k} - Ce^{-kt}$$

Constant C determined from the initial condition $t = 0, v = v_0$:

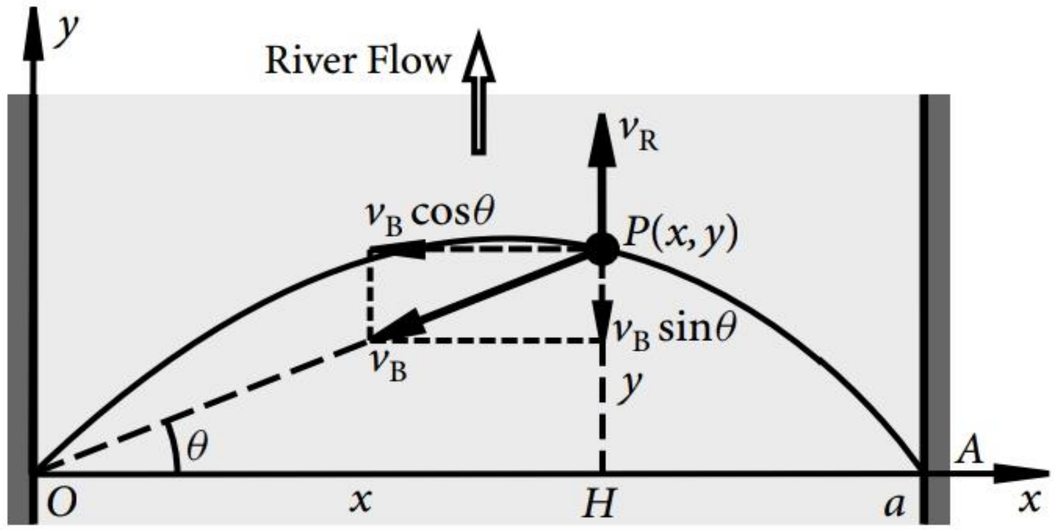
$$v_0 = \frac{g}{k} - Ce^0 \quad , \qquad\qquad \therefore C = \frac{g}{k} - v_0$$

When $t \to \infty, v = v_E$; $v_E = \frac{g}{k}$; $C = v_E - v_0$. Hence the velocity is given by $v = v_E - (v_E - v_0)e^{-kt}$

$$\therefore \frac{v - v_E}{v_0 - v_E} = e^{-kt}$$

FERRY BOAT

A ferry boat is crossing a river of width a from point A to point O as shown in the following figure. The boat is always aiming toward the destination O. The speed of the river flow is constant v_R and the speed of the boat is constant v_B. Determine the equation of the path traced by the boat



Suppose that, at time t, the boat is at point P with coordinates (x, y). The velocity of the boat has two components the velocity of the boat v_B relative to the river flow (as if the river is not flowing), which is pointing toward the origin O or along line PO, and the velocity of the river v_R in the y direction.

Decompose the velocity components v_B and v_R in the x and y directions

$$v_x = -v_B \cos\theta \quad , v_y = v_R - v_B \sin\theta$$

From $\triangle$ OHP, $\cos\theta = \frac{OH}{OP} = \frac{x}{\sqrt{x^2+y^2}}$ $\sin\theta = \frac{PH}{OP} = \frac{y}{\sqrt{x^2+y^2}}$

Hence the equations of motions are given by

$$v_x = \frac{dx}{dt} = -v_B \frac{x}{\sqrt{x^2+y^2}} \quad v_y = \frac{dy}{dt} = v_R - v_B \frac{y}{\sqrt{x^2+y^2}}$$

Since only the equation between x and y is sought, variable t can be eliminated by dividing these two equations

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v_R - v_B \frac{y}{\sqrt{x^2+y^2}}}{-v_B \frac{x}{\sqrt{x^2+y^2}}} = \frac{k\sqrt{x^2+y^2} - y}{x}, \qquad k = \frac{v_R}{v_B}$$

$$= -k\sqrt{1+\left(\frac{y}{x}\right)^2} + \frac{y}{x} \qquad \text{(Homogeneous D.E)}$$

Let $u = \frac{y}{x}$ $or\ y = xu, \frac{dy}{dx} = u + x\frac{du}{dx}$. Hence, the equations becomes

$$u + x\frac{du}{dx} = -k\sqrt{1+u^2} + u\,. \qquad \text{(Variable separable)}$$

The general solution is

$$\int \frac{du}{\sqrt{1+u^2}} = -k\int \frac{dx}{x} \qquad \Longrightarrow \ln\left(u + \sqrt{1+u^2}\right) = -klnx + lnC$$

$$\therefore u + \sqrt{1+u^2} = Cx^{-k}$$

Replacing u by the original variables yields

$$\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} = Cx^{-k} \Longrightarrow \sqrt{x^2+y^2} = Cx^{1-k} - y$$

Squaring both sides leads to

$$x^2 + y^2 = C^2x^{2(1-k)} - 2Cx^{1-k}y + y^2$$

$$x^2 = C^2x^{2(1-k)} - 2Cx^{1-k}y$$

The constant C is determined by the initial condition t=0, x=a, y=0:

$$a^2 = C^2x^{2(1-k)} - 0 \Longrightarrow C = a^k$$

Hence, the equation of the path is

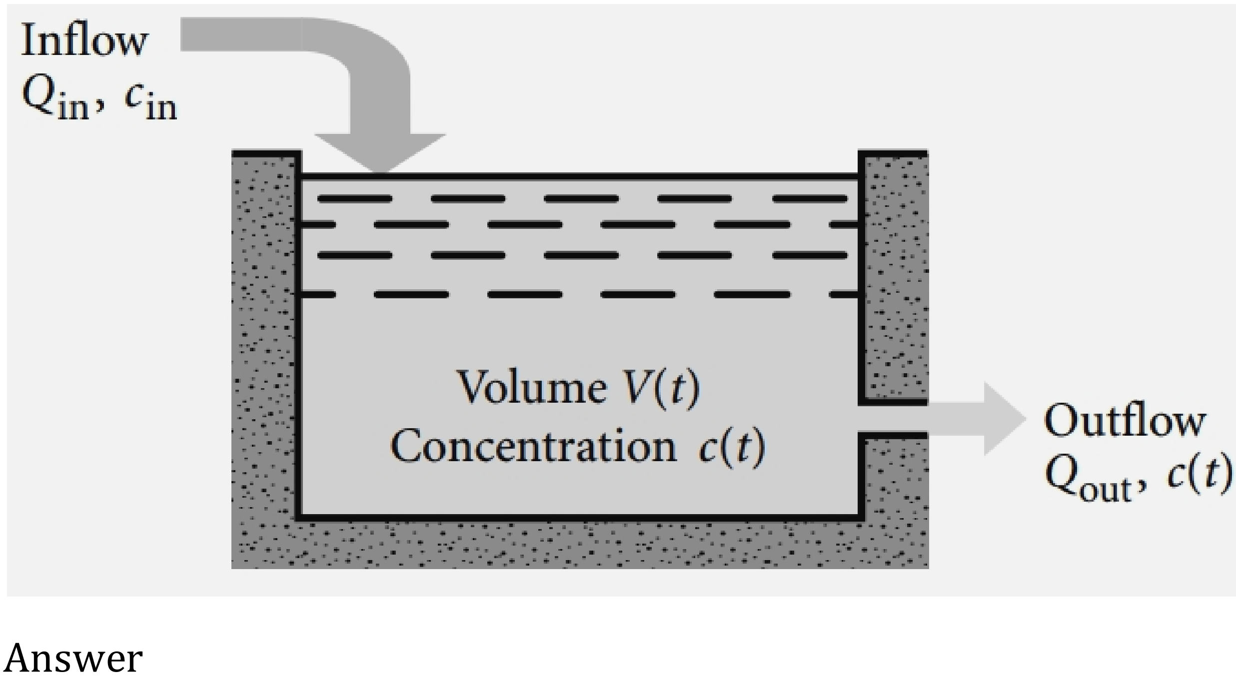
$$y = \frac{1}{2Cx^{1-k}}\left[C^2x^{2(1-k)} - x^2\right] = \frac{1}{2}\left(a^k x^{1-k} - a^{-k}x^{1+k}\right)$$

$$y = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k}\right]$$

<u>RESERVOIR POLLUTION</u>

A reservoir initially contains polluted water of volume V_0 $(m^3)$with a pollutant concentration in percentage being c_0. In order to reduce $c(t)$, which is the pollutant concentration in the reservoir at time t, it is arranged to have in flow and out flow of water at the rates of Q_0 andQ_0 $(\frac{m^3}{day})$, respectively, as shown in the following figure. Unfortunately, the inflowing water is also polluted, but to a lower extent ofc_{in}. Assume that the out flowing water is perfectly mixed.

Setup the differential equation governing the pollutant $c\ (t)$



Answer

At time t,

Volume $V(t) = V_o + (Q_{in} - Q_{out})t,$

Pollution concentration $c(t)$

Amount of pollutant $= V(t)c(t) = [V_0 + (Q_{in} - Q_{out})t]c$

At time$t + \Delta t$,

Volume $V(t + \Delta t) = V_o + (Q_{in} - Q_{out})(t + \Delta t)$,

Pollutant concentration $c(t + \Delta t) = c(t) + \Delta c$,

Amount of pollutant$= V(t + \Delta t)c(t + \Delta t)$

$$= [V_o + (Q_{in} - Q_{out})(t + \Delta t)](c + \Delta c)$$

Inflow pollutant $= Q_{in}\Delta t c_{in}$

Out flow pollutant. $= Q_{out}\Delta t c$

Since (Amount of pollutant at $t + \Delta t$)= (Amount of pollutant at t) +[(Inflow pollutant)−(Out flow pollutant)

$$\therefore [V_o + (Q_{in} - Q_{out})(t + \Delta t)](c + \Delta c)$$
$$= [V_0 + (Q_{in} - Q_{out})t]c + Q_{in}\Delta t c_{in} - Q_{out}\Delta t c$$

Expanding yields,

$$[V_0 + (Q_{in} - Q_{out})t]c + [V_0 + (Q_{in} - Q_{out})t]\Delta c + (Q_{in} - Q_{out})\Delta t c + (Q_{in} - Q_{out})\Delta t \Delta c$$
$$= [V_0 + (Q_{in} - Q_{out})t]c + Q_{in}\Delta t c_{in} - Q_{out}\Delta t c$$

Neglecting higher order term$\Delta t \Delta c$, dividingby Δt, and simplifying lead to

$$[V_0 + (Q_{in} - Q_{out})t]\frac{\Delta c}{\Delta t} + Q_{in}c = Q_{in}c_{in}$$

Taking the limit $\Delta t \to 0$ results in the differential equation

$$[V_0 + (Q_{in} - Q_{out})t]\frac{dc}{dt} + Q_{in}c = Q_{in}c_{in}$$

NEWON'S LAW OF COOLING

According to this law, the temperature of a body changes at a rate t which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.

If θ_0 is the temperature of the surroundings and θ that of the body at any time t, then

$\frac{d\theta}{dt} = -k(\theta - \theta_0)$, Where k is a constant

This is a first order linear equation.

Example: A body originally at 80℃ cools down to 60℃ in 20 minutes,the temperature of the air being 40℃. What will be the temperature of the bodyafter 40 minutes from the original?

Answer: If θ be the temperature of the body at any time t, then

$\frac{d\theta}{dt} = -k(\theta - 40)$, Where k is a constant

Integrating $\int \frac{d\theta}{\theta-40} = -k \int dt + \log c$ where c is a constant

$$\log(\theta - 40) = -kt + \log c \; ; \qquad (1)$$

I.e. $(\theta - 40) = ce^{-kt}$ (2)

When $t = 0$, $\theta = 80$℃ from (1) ; $\log(80 - 40) = -0 + \log c$

$$\therefore c = 40$$

And when $t = 20min$, $\theta = 60$℃

From equation (2) we get $20 = ce^{-20t}$

$20 = 40e^{-20t}$

$\therefore\ k = \frac{1}{20}\log 2$

Thus equation (2) becomes$(\theta - 40) = 40e^{(-\frac{1}{20}\log 2)t}$

When t=40 min, $\theta = 40 + 40e^{-\,2\log 2}$

$$= 40 + 40e^{\log\left(\frac{1}{4}\right)}$$

$$= 40 + 40 * \frac{1}{4} = 50°\text{C}$$

$$\theta = 50°\text{C}$$

Law of Radioactive Decay

When a radioactive material undergoes α, β or γ-decay, the number of nuclei undergoing the decay, per unit time, is proportional to the total number of nuclei in the sample material. So, if N = total number of nuclei in the sample and ΔN = number of nuclei that undergo decay in time Δt then,

$$\frac{N}{t}\ \ \alpha\ \ N$$

$$\text{Or}\ \frac{\Delta N}{\Delta t} = \lambda\, N \qquad (1)$$

Where λ = radioactive decay constant or disintegration constant. Now, the change in the number of nuclei in the sample is, $dN\ =\ -\ \Delta N$ in time Δt. Hence, the rate of change of N (in the limit $\Delta t\ \rightarrow 0$) is,

$$\frac{dN}{dt} = -\lambda N \qquad (\,first\ order\ differntial\ equation)$$

$$\text{Or}\frac{dN}{N} = -\lambda\ t$$

Now, integrating both the sides of the above equation, we get,

$$\int_{N_O}^{N} \frac{dN}{N} == -\lambda \int_{t_o}^{t} dt \qquad (2)$$

Or $$, ln\, N - \, ln\, N0 \; = - \; \lambda \; (t - \; t0) \qquad (3)$$

Where, N_0 is the number of radioactive nuclei in the sample at some arbitrary time t_0 and N is the number of radioactive nuclei at any subsequent time t. Next, we set $t_0 = 0$ and rearrange the above equation (3) to get,

$$\ln\left(\frac{N}{N_O}\right) = -\lambda t$$

Or $$N = N_o e^{-\lambda t} \qquad (4)$$

Equation (4) is the Law of Radioactive Decay.

<u>Half life</u>. $\left(T_{\frac{1}{2}}\right)$

The term half-life is defined as the time it takes for one-half of the atoms of a radioactive material to disintegrate (decay).

<u>Decay constant (λ)</u>

The decay constant (units: s^{-1}) of a radioactive nuclide is its probability of decay per unit time. The decay constant relates to the half-life of the nuclide $T_{1/2}$ through

$$\lambda = \frac{ln2}{T_{\frac{1}{2}}} \qquad i.e\;\; \lambda = \frac{0.693}{T_{\frac{1}{2}}}$$

Radiocarbon Dating

Radiocarbon dating is a method of estimating the age of organic material. It was developed right after World War II by Willard F. Libby and co-workers, and it has provided a way to determine the ages of different materials in archaeology, geology, geophysics, and other branches of science. Some examples of the types of material that radiocarbon can determine the ages of are wood, charcoal, marine and freshwater shell, bone and antler, and peat and organic-bearing sediments. Age determinations can also be obtained from carbonate deposits such as calcite, dissolved carbon dioxide, and carbonates in ocean, lake, and groundwater sources.

Carbon 14 Dating

Archaeologists use the exponential, radioactive decay of carbon 14 to estimate the death dates of organic material. The stable form of carbon is carbon 12 and the radioactive isotope carbon 14 decays over time into nitrogen 14 and other particles. Carbon is naturally in all living organisms and is replenished in the tissues by eating other organisms or by breathing air that contains carbon. At any particular time all living organisms have approximately the same ratio of carbon 12 to carbon 14 in their tissues. When an organism dies it ceases to replenish carbon in its tissues and the decay of carbon 14 to nitrogen 14 changes the ratio of carbon 12 to carbon 14. We can compare the ratio of carbon 12 to carbon 14 in dead material to the ratio when the organism was alive to estimate the date of its death. Radiocarbon dating can be used on samples of bone, cloth, wood and plant fibres.

The half-life of a radioactive isotope describes the amount of time that it takes half of the isotope in a sample to decay. In the case of radiocarbon dating, the half-life of carbon 14 is 5,730 years. This half life is a relatively small number, which means that carbon 14 dating is not particularly helpful for very recent deaths and deaths more than 50,000 years ago. After 5,730 years, the amount of carbon 14 left in the body is half of the original amount. If the amount of carbon 14 is halved every 5,730 years, it will not take very long to reach an amount that is too small to analyze. When finding the age of an organic organism we need to consider the half-life of carbon 14 as well as the rate of decay (λ)

Question :

A fossil is found that has 35% carbon 14 compared to the living sample. How old is the fossil?

Answer: We consider the decay of carbon 14 to find the answer.

By radioactive decay law we know

$$\frac{dN}{dt} = \lambda N_O$$

By integrating we get $N = N_o e^{-\lambda t}$

$$t = \frac{\ln\left(\frac{N}{N_O}\right)}{-\lambda} \qquad (1)$$

Here $N_o = Amount\ of\ carbon\ 14\ in\ the\ living\ sample$

$N = Amount\ of\ carbon\ 14\ \ left\ \ in\ the\ fossil(after\ decay)$
$= 35\%\ of\ N_0 = 0.35N_O$

$$\lambda = Decay\ contant = \frac{0.693}{T_{\frac{1}{2}}}(years)^{-1}$$

Where $T_{\frac{1}{2}}$ is the half-life of the isotope carbon -14,

$$t = \left(\frac{\ln\left(\frac{N}{N_O}\right)}{-0.693} \right) . T_{\frac{1}{2}} \qquad (2)$$

$T_{\frac{1}{2}} = 5730\ years$

t is the age of the fossil (or the date of death).

then we can substitute values into our equation.

$$t = \left(\frac{ln\left(\frac{0.35N_O}{N_o}\right)}{-0.693} \right) \times 5730$$

$$t = \left(\frac{ln(0.35)}{-0.693} \right) \times 5730$$

$$t = (1.5149) \times 5730$$

$$t = 8680\ years$$

So, the fossil is 8,680 years old, meaning the living organism died 8,680 years ago

The SIR epidemic disease model

The SIR model, first published by Kermack and McKendrick in 1927, is undoubtedly the most famous mathematical model for the spread of an infectious disease. Here, people are characterized into three classes: susceptible S, infective I and removed R. Removed individuals are no longer susceptible nor infective for whatever reason; for example, they have recovered from the disease and are now immune, or they have been vaccinated, or they have been isolated from the rest of the population, or perhaps they have died from the disease. The model may be diagrammed as

$$S \rightarrow I \rightarrow R$$

This model is an appropriate one to use under the following assumptions

1) The population is fixed.

2) The only way a person can leave the susceptible group is to become infected. The only way a person can leave the infected group is to recover or die from the disease. Once a person has removed from the infected group, the person received immunity.

3) Age, sex, social status, and race do not affect the probability of being infected.

4) There is no inherited immunity.

5) The member of the population mix homogeneously (have the same interactions with one another to the same degree).

The model starts with some notations:

S(t) is the number of susceptible individuals at time t

I(t) is the number of infected individuals at time t

R(t)is the number of removed individuals at time t

N is the total population size.

The assumptions lead us to a set of differential equations.

$$\frac{dS}{dt} = -\frac{B}{N}SI \qquad (1)$$

$$\frac{dI}{dt} = \frac{B}{N}SI - aI \qquad (2)$$

$$\frac{dR}{dt} = aI \qquad (3)$$

"B" and "a" are the two most important elements in this model.

- "B" controls the contact rate or the average number of contacts that turn out to be infected for every infected individual per day,i.e ,The parameter "B" indicates the rate at which susceptible individuals get infected .

- "a" controls the removal rate or the proportion of people recovering/dying/isolating on a daily basis, i.e. The parameter "a" indicates the rate at which infected individuals recover (or die).

$$\frac{dS}{dt} = -\frac{b}{N}SI(1)$$

$$\frac{dI}{dt} = \frac{b}{N}SI - aI(2)$$

$$\frac{dR}{dt} = aI(3)$$

When number of infectives start to grow from initial value I_0,then we will have spread of disease through a population

i.e. if $\frac{dI}{dt}$ is positive, the spread occurs

Spread occurs when , $0 < \frac{dI}{dt}$

From equation (2) $0 < \frac{dI}{dt} = \frac{B}{N}SI - aI$

$$0 < \frac{dI}{dt} = I\left(\frac{B}{N}S - a\right)(4)$$

We know that $S < S_0$ (S_0 Is initial number of susceptible and S is number of susceptible at time t)

$$I\left(\frac{B}{N}S - a\right) < I\left(\frac{B}{N}S_0 - a\right)$$

Applying this in equation (4)

$$0 < \frac{dI}{dt} = I\left(\frac{B}{N}S - a\right) < I\left(\frac{B}{N}S_0 - a\right)$$

$$0 < \frac{dI}{dt} < I\left(\frac{B}{N}S_0 - a\right)$$

$$0 < I\left(\frac{B}{N}S_0 - a\right)$$

Since I is positive, $0 < \left(\frac{B}{N}S_0 - a\right)$

$$a < \frac{B}{N}S_0$$

$$1 < \frac{BS_0}{Na} \qquad (S_0 \sim N) \ \ (Initially\ all\ are\ susceptible)$$

$$1 < \frac{B}{a} = R_0$$

$R_0 = \frac{B}{a}$, is the Basic Reproduction Number (the number of cases that are expected to occur on average homogeneous population)

Here we can see that the spread occurs when $1 < R_0$

Therefore we can conclude that if $R_0 > 1$ the spread will happen.

- R_0 is the average number of people infected from one otherperson. If it is high, the probability of pandemic is also higher.
- If $R_0 < 1$
  The disease is expected to stop spreading.
- If $R_0 = 1$
  The disease spread is stable or endemic, and the number of infections is not expected to increase or decrease.
- If $R_0 > 1$
  The disease is expected to increasingly spread in the absence of intervention.

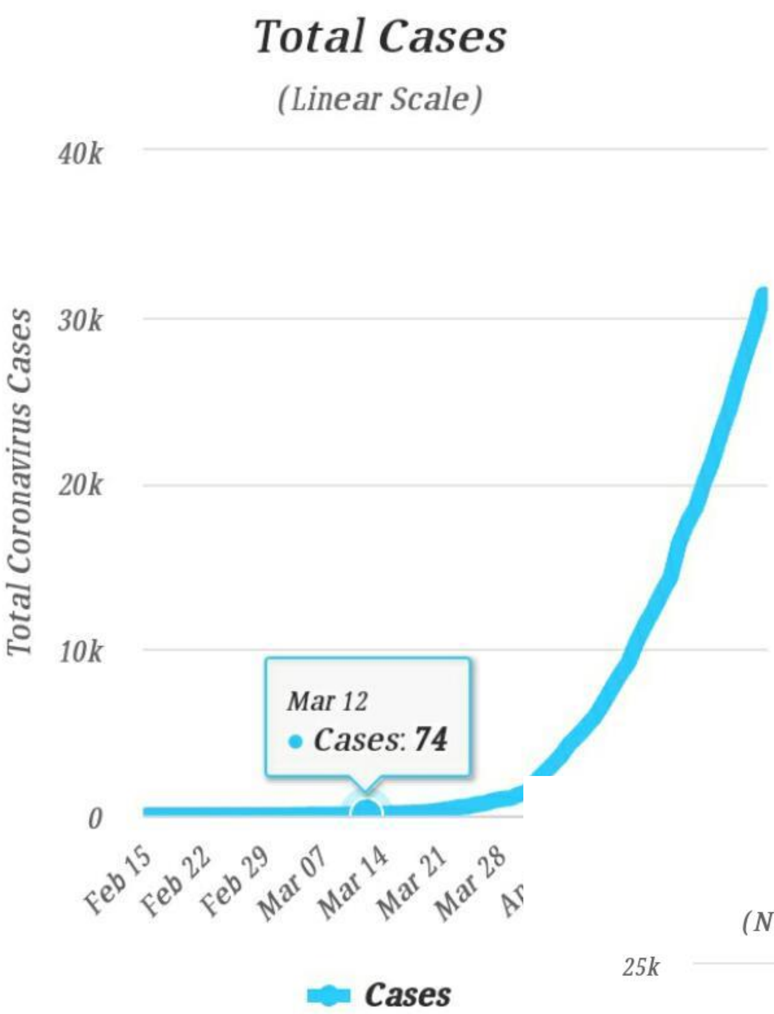
To stop the spreading of disease: we have to make $R_0 = \frac{B}{a} < 1$

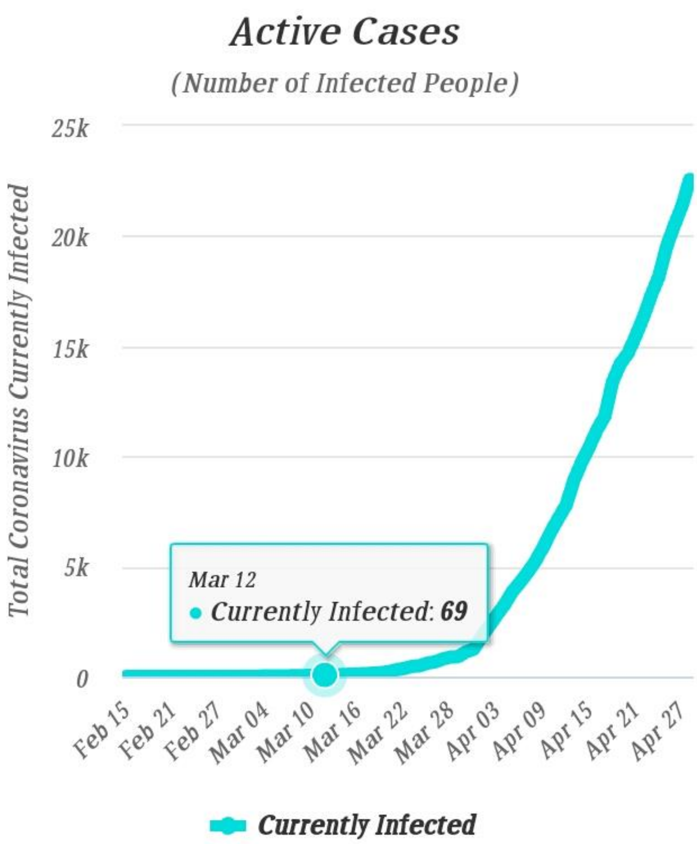
In the recent public policy terms,

- "*B*" is the parameter that controls your **social distancing** (*lower B, fewer contacts*)
- "*a*" controls your **self-quarantine strategy** (*higher a, lower period of infectivity*)
- Together, $R_o = (B/a) \propto$ *the number of secondary cases for every primary case*

Case study India: Corona virus (CoV) is a newly discovered infectious disease with rapid spread. We are still in the early stages of the COVID-19 outbreak and there is great uncertainty about the characteristics of this virus. Right now, many places seeing Covid-19 transmission are following an exponential growth trajectory. That is, the rate of the spread of the infection is proportional to the number of people infected. Each infected person is expected to infect a certain number of people

DATE	TOTAL CASES	ACTIVE CASES (I)
10 March	62	58
11 March	62	58
12 March	74	69
13 March	82	70
14 March	100	88
15 March	114	90


Total Cases
(Linear Scale)
40k
30k
20k
10k
0
Total Coronavirus Cases
Mar 12
Cases: 74
Feb 15
Feb 22
Feb 29
Mar 07
Mar 14
Mar 21
Mar 28
Cases


Active Cases
(Number of Infected People)
25k
20k
15k
10k
5k
0
Total Coronavirus Currently Infected
Mar 12
Currently Infected: 69
Feb 15
Feb 21
Feb 27
Mar 04
Mar 10
Mar 16
Mar 22
Mar 28
Apr 03
Apr 09
Apr 15
Apr 21
Apr 27
Currently Infected

Date	Active cases (I)	Removed cases(R)	Susceptible cases(S)	Change in S	B	Change in R	a
11 march	58	4	1,380,004,323	0	0	0	0
12 march	69	5	1,380,004,311	12	0.2068	1	0.0172
13 march	70	12	1,380,004,303	8	0.1159	7	0.1014
14 march	88	12	1,380,004,285	18	0.2571	0	0
15 march	99	15	1,380,004,271	14	0.1590	3	0.0340

TO FIND*B* and *a*

The equations used are :

$$\frac{\Delta S}{\Delta t} = -\frac{B}{N} SI \qquad (1)$$

$$\frac{\Delta R}{\Delta t} = aI \qquad (2)$$

❖ E.g. to find B and a for the date 12 March 2020

- From equation (1)

$$\frac{\Delta S}{\Delta t} = \frac{S_{12} - S_{11}}{\Delta t} = -\frac{B}{N} S_{11} I_{11} \qquad (3)$$

$N = 1{,}380{,}004{,}385$

$S_{11} = 1380004323$, $S_{12} = 1380004311$; $\Delta S = -12$

$\Delta t = 1, \quad I_{11} = 58$

Substituting the values in Equation (3)

$$\frac{1380004311 - 1380004323}{1} = -\frac{B}{1380004385} \times 1380004323 \times 58$$

$B = 0.2068$

- From Equation (2)

$$\frac{\Delta R}{\Delta t} = \frac{R_{12} - R_{11}}{\Delta t} = aI_{11}(4)$$

$N = 1380004385$, $R_{11} = 4$, $R_{12} = 5$; $\Delta R = 1$

$\Delta t = 1$, $I_{11} = 58$

Substituting the values in Equation (4)

$$\frac{5 - 4}{1} = a \times 58$$

$a = 0.0172$

$\therefore$ $B = 0.2068$ and $a = 0.0172$ for the date 12 March 2020

Consider average of B and a over 5 days

$$B_{avg} = \frac{0.2068 + 0 + 0.1159 + 0.2571 + 0.1590}{5} = \frac{0.7388}{5} \quad ;$$

$$a_{avg} = \frac{0.0172 + 0 + 0.1014 + 0 + 0.0340}{5} = \frac{0.1526}{5}$$

$$R_0 = \frac{B_{avg}}{a_{avg}} = \frac{\frac{0.7388}{5}}{\frac{0.1526}{5}} = \frac{0.7388}{0.1526} = 4.8$$

THEREFORE $R_0 = 4.8 > 1$ OUTBREAK WILL HAPPEN.

- To decrease the spread :$R_0 < 1$

  - Measures to make $R_0 = \frac{B}{a} < 1$

    1. We have to decrease B.

       For decreasing 'B' ,it can be done by stricter government border control.

       Developing vaccines can also decreases the value of 'B' significantly. But this definitely takes quite a bit of time to do.

    2. We have to increase ‘a'

       For increasing 'a' however, there is not much that we can do about other than developing better medicine to treat patients.

- Staying at home as much as possible decreases the chance of infections happening in the first place. Of course observing personal hygiene is also important in decreasing infection.

As the spread of COVID-19 continues, communities are being asked to reduce close contact between people. This is called social distancing, and it's an important and effective way to slow down the spread of this virus. Eventually, the chance of an infected person contacting a susceptible person becomes low enough that the rate of infection decreases, leading to fewer cases and eventually, the end of the viral spread. As there is currently no vaccine or specific drug for COVID-19, the only ways we can reduce transmission is through good hygiene, isolating suspected cases, and by social distancing measures such as cancelling large events and closing schools.

Summary

- Epidemic can be represented by compartmental ODE models
- Even the simplest epidemiological models require computer algorithms to estimate prevalence profiles
- But criteria for invasion and for equilibrium conditions can be derived analytically
- The basic reproductive ratio R_0 is a key epidemiological measure affecting criteria for invasion extinction and size of the epidemic
- An infection experiences deterministic extinction if $R_0 < 1$

CHAPTER THREE

Applications of Second-Order Differential Equations

Second-order linear differential equations have a variety of applications in science and engineering. In this section we explore the vibration of springs and LCR circuits

Vibrating Springs

We consider the motion of an object with mass at the end of a spring that is either vertical (as in Figure 1) or horizontal on a level surface (as in Figure 2).

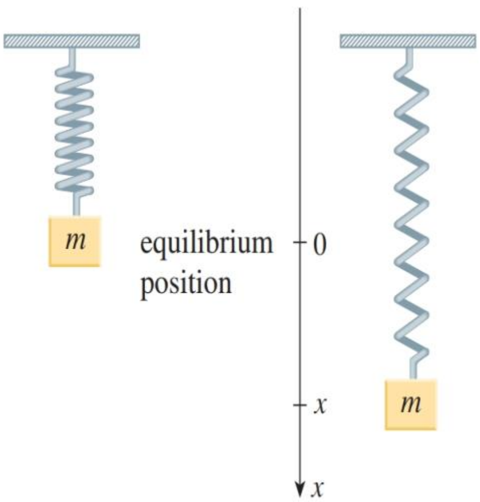


FIGURE 1

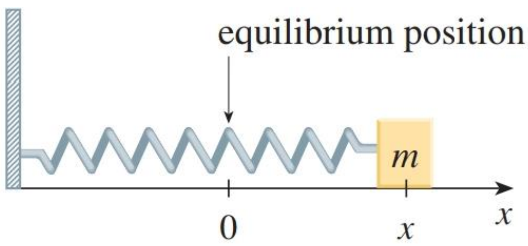


FIGURE 2

According to Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x

Restoring Force $= -kx$

Where k is a positive constant (called the spring constant). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$m\frac{d^2x}{dt^2} = -kx \text{ or } m\frac{d^2x}{dt^2} + kx = 0 \qquad (1)$$

This is a second-order linear differential equation. Its auxiliary equation is $mr^2 + k = 0$

with roots $r = \pm\omega i$, where $\omega = \sqrt{\frac{k}{m}}$. Thus, the general solution is

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t$$

Which can also be written as

$$x(t) = c_1 \cos(\omega t + \delta)$$

Where $\omega = \sqrt{\frac{k}{m}}$(frequency)

$A = \sqrt{{c_1}^2 + {c_2}^2}$(Amplitude)

$\cos\delta = \frac{c_1}{A} \sin\delta = -\frac{C_2}{A}$ ($\delta$is the phase angle)

[NOTE $\quad x(t) = c_1 \cos(\omega t + \delta) \Longleftrightarrow$

$x(t) = A[\cos\omega t \cos\delta - \sin\omega t \sin\delta] \quad \Longleftrightarrow$

$x(t) = A(\frac{c_1}{A}\cos\omega t + \frac{c_2}{A}\sin\omega t)$ Where $\cos\delta = \frac{c_1}{A}$ *and* $\sin\delta = -\frac{C_2}{A}$

$x(t) = c_1 \cos\omega t + c_2 \sin\omega t$ and

$(\cos\delta)^2 + (\sin\delta)^2 = 1;\ c_1^2 + c_2^2 = A$]

This type of motion is called simple harmonic motion.

Question 1 : A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6N is required to maintain it stretched to a length of 0.7m. If the spring is stretched to a length of 0.7m and then released with initial velocity 0, find the position of the mass at anytime t .

Answer: From Hooke's Law, the force required to stretch the spring is

$k(0.2) = 25.6 \qquad (x = 0.7 - 0.5 = 0.2m)$

$$k = \frac{25.6}{0.2} = 128;\ m = 2kg;\ \ \omega = \sqrt{\frac{128}{2}} = 8s^{-1}$$

Using values of k and m in Equation 1, we have

$$2\frac{d^2x}{dt^2} + 128x = 0$$

As in the earlier general discussion, the solution of this equation is

$$x(t) = c_1 \cos 8t + c_2 \sin 8t \qquad (2)$$

We are given the initial condition that$x(0) = 0.2$. But, from Equation 2,$x(0) = c_1 \quad \therefore c_1 = 0.2$.

Differentiating Equation 2, we get$x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t$

Since the initial velocity is given as $x'(0) = 0$, we have $c_2 = 0$ and so the solution is

$$x(t) = 0.2 \cos 8t$$

<u>Damped Vibration</u>s

We next consider the motion of a spring that is subject to a frictional force (in the case of the horizontal spring of Figure 2) or a damping force (in the case where a vertical spring moves through a fluid as in Figure 3). An example is the damping force supplied by a shock absorber in a car or a bicycle.

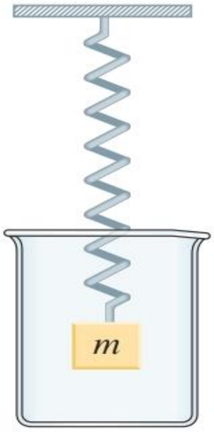


FIGURE 3

We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. (This has been confirmed, at least approximately, by some physical experiments.) Thus

Damping force $= -c\frac{dx}{dt}$

where is a positive constant, called the Damping constant. Thus, in this case, Newton’s Second Law gives

$$m\frac{d^2x}{dt^2} = restoring\ force + damping\ force = -kx - c\frac{dx}{dt}$$

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0} \qquad (3)$$

Equation (3) is a second order linear differential equation and its auxiliary equation is$mr^2 + cr + k = 0$. The roots are

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \qquad (4)$$

We need to discuss three cases.

- CASE I $c^2 - 4mk > 0$ (overdamping) In this case r_1 and r_2 are distinct real roots and the solution is
  $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
  Since c, m and k are all positive, we have $\sqrt{c^2 - 4mk} < 0$, so the roots r_1 and r_2 given by Equations 4 must both be negative. This shows that $x \to 0 \ as \ t \to \infty$. Typical graphs of x as a function of t are shown in Figure 4. Notice that oscillations do not occur. (It's possible for the mass to pass through the equilibrium position once, but only once.) This is because $c^2 > 4mk$ means that there is a strong damping force (high-viscosity oil or grease) compared with a weak spring or small mass.
- CASE II $c^2 - 4mk = 0$ (critical damping) This case corresponds to equal roots $r_1 = r_2 = -\frac{c}{2m}$ and the solution is given by
  $$x = (c_1 + c_2 t)e^{-\left(\frac{c}{2m}\right)t}$$
  It is similar to Case I, and typical graphs resemble those in Figure 4, but the damping is just sufficient to suppress vibrations. Any decrease in the viscosity of the fluid leads to the vibrations of the following case.

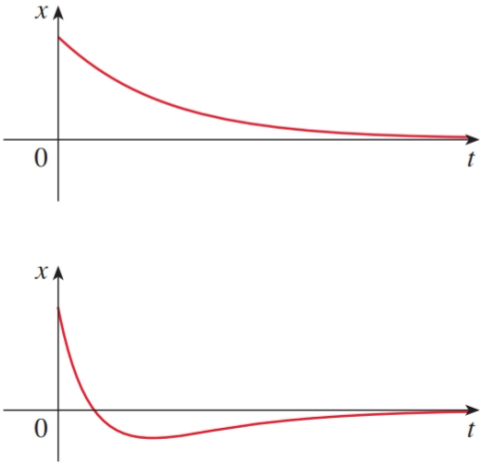


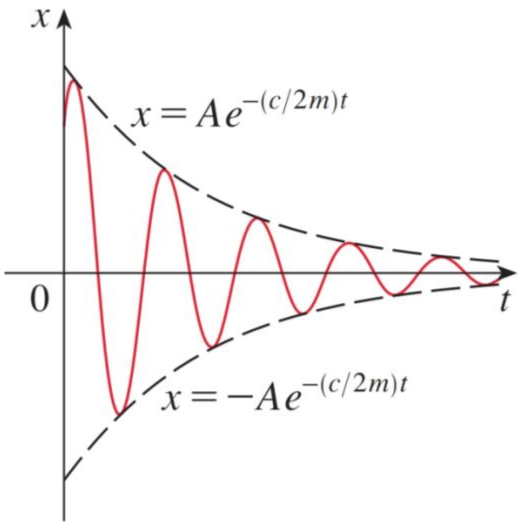
FIGURE 4
Overdamping

- CASE III $c^2 - 4mk < 0$ (underdamping) Here the roots are complex:

  $$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = -\left(\frac{c}{2m}\right) \pm \omega i \;\; where\ \omega = \frac{\sqrt{4mk - c^2}}{2m}$$

  The solution is given by$x = e^{-\left(\frac{c}{2m}\right)t}(c_1 \cos \omega t + c_2 \sin \omega t)$

  We see that there are oscillations that are damped by the factor$e^{-\left(\frac{c}{2m}\right)t}$. Since $c > 0$ and $m > 0$, we have $-\left(\frac{c}{2m}\right) < 0$ so $e^{-\left(\frac{c}{2m}\right)t} \to 0\ as\ t \to \infty$ .This implies that $x \to 0\ as\ t \to \infty$; that is, the motion decays to 0 as time increases. A typical graph is shown in Figure 5.



EXAMPLE 2 Suppose that the spring of Example 1 is immersed in a fluid with damping constant $c = 40\ kg/s$. Find the position of the mass at any time$t$ if it starts from the equilibrium position and is given a push to start it with an initial velocity of $0.6m/s$.

SOLUTION

From Example 1 the mass is $m = 2kg$ and the spring constant is $k = 128kg/s^2$, so the differential equation (3) becomes

$$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0$$

Or

$$\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 64x = 0$$

The auxiliary equation is $r^2 + 20r + 64 = (r + 4)(r + 16) = 0$ with roots -4 and-16 , so the motion is overdamped and the solution is

$$x = c_1e^{-4t} + c_2e^{-16t}$$

We are given that $x(0) = 0$, so $c_1 + c_2 = 0$. Differentiating, we get

$$x'(t) = -4c_1e^{-4t} - 16c_2e^{-16t}$$

$$so \;\; x'(0) = -4c_1 - 16c_2 = 0.6 \qquad \{x'(0) = initial\ velocity\}$$

Since$c_2 = -c_1$, this gives$12c_1 = 0.6$ or $c_1 = 0.05$. Therefore $x = 0.05(e^{-4t} - e^{-16t})$

Figure 6 shows the graph of the position function for the overdamped motion in Example 2.

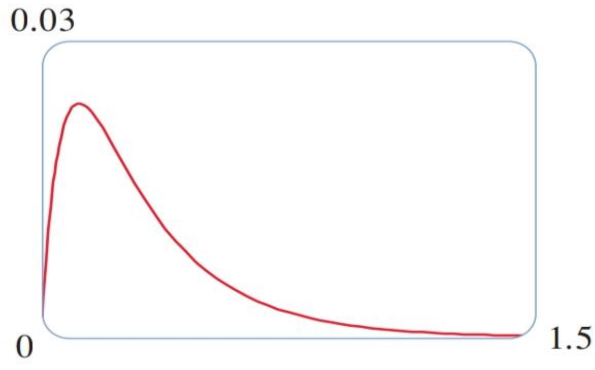


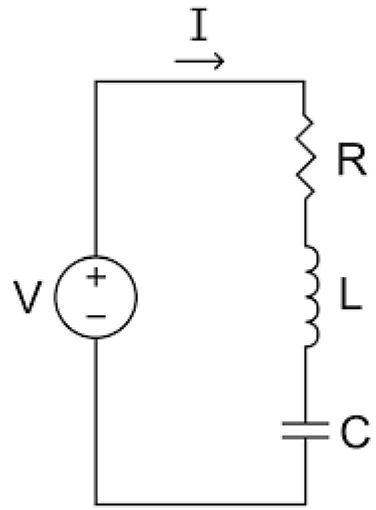
FIGURE 6

OSCILLATORY ELECTRICAL CIRCUIT

LCR Circuits

Consider an electrical circuit containing an inductance L, capacitance C and resistance R.

Let i be the current and q the charge in the condenser plate at any time t,



Now consider the discharge of a condenser C through an inductance L and the resistance R.
Since the voltage drop across L, C and R are respectively

$$L\frac{d^2q}{dt^2}, \frac{q}{C}, and\ R\frac{dq}{dt}$$

Therefore by Kirchhoff's first law, we have

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

$$\frac{R}{L} = 2\lambda\ and\ \frac{1}{LC} = \mu^2.$$

λ denotes additional force of resistance

we have

$$\frac{d^2q}{dt^2} + 2\lambda\frac{dq}{dt} + \mu^2\frac{q}{C} = 0 \qquad (1)$$

This is a second order differential equation.
This equation is same as the equation of the mass m on a spring with a damper.

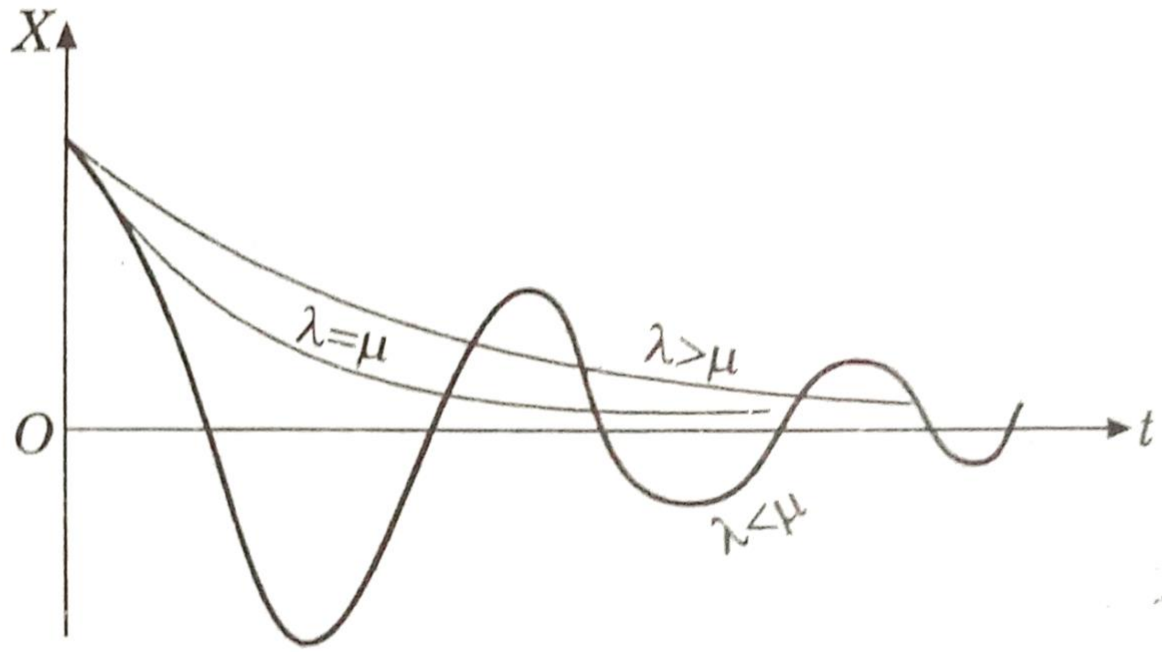
Therefore equation (1) has the same solution as for the mass m on a spring with a damper

Auxiliary equation of (1) is $r^2 + 2\lambda r + \mu^2 = 0$

the roots of the auxiliary equation are

$$r_1 = \frac{-2\lambda + \sqrt{4\lambda^2 - 4\mu^2}}{2} r_2 = \frac{-2\lambda - \sqrt{4\lambda^2 - 4\mu^2}}{2}$$

$$\therefore r_1 = -\lambda + \sqrt{\lambda^2 - \mu^2} r_2 = -\lambda - \sqrt{\lambda^2 - \mu^2}$$



Case 1 When $\lambda > \mu$, the roots of the auxiliary equation are real and distinct (say $r_{1,} r_2$).

Therefore, the solution of (1) is

$$q = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Which shows that q is always positive and decreases to zero as $t \to \infty$

The restoring force, in this case, is so great that the motion is non-oscillatory and is, therefore, referred to as over-damped or dead-beat motion.

Case2 When $\lambda = \mu$, the roots of the auxiliary equation are real and equal,

$$(r_1 = r_2 = -\lambda)$$

Therefore, the general solution of (1) becomes

$$q = (c_1 + c_2 t) e^{-\lambda t}$$

Which shows that q is always positive and decreases to zero as $t \to \infty$.

The nature of motion is similar to that of the previous case and is called the critically damped motion for it separates the non-oscillatory motion of case1 from the most interesting oscillatory motion of case 3.

Case 3 When $\lambda < \mu$ the roots of the auxiliary equation are imaginary, i.e.

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = -\lambda \pm i\omega \; where \; \omega^2 = \mu^2 - \lambda^2$$

Therefore, the solution of (1) becomes

$$q = e^{-\lambda t}(c_1 \cos \omega t + c_2 \sin \omega t) \qquad (2)$$

Here the presence of the trigonometric factor in (2) shows that the motion is oscillatory, having

a)the variable amplitude which decreases with time

b)the periodic time $T = \frac{2\pi}{\omega}$

But the periodic time of free oscillations is$T_0 = \frac{2\pi}{\mu}$

$[\;i.e. when \; \lambda = 0 \; (no \; additional \; force \; of \; resistance)]$

As $\omega = \sqrt{\mu^2 - \lambda^2} < \mu$

$$\frac{2\pi}{\omega} > \frac{2\pi}{\mu}$$

$i.e.\;\; T > T_0$

This shows that the effect of damping is to increase the period of oscillation and the motion ultimately dies away. Such a motion is termed as damped oscillatory motion.

Thus the charging or discharging of a condenser through the resistance R and an inductance L is an electrical analogue of the damped oscillations of mass m on a spring.

CHAPTER FOUR

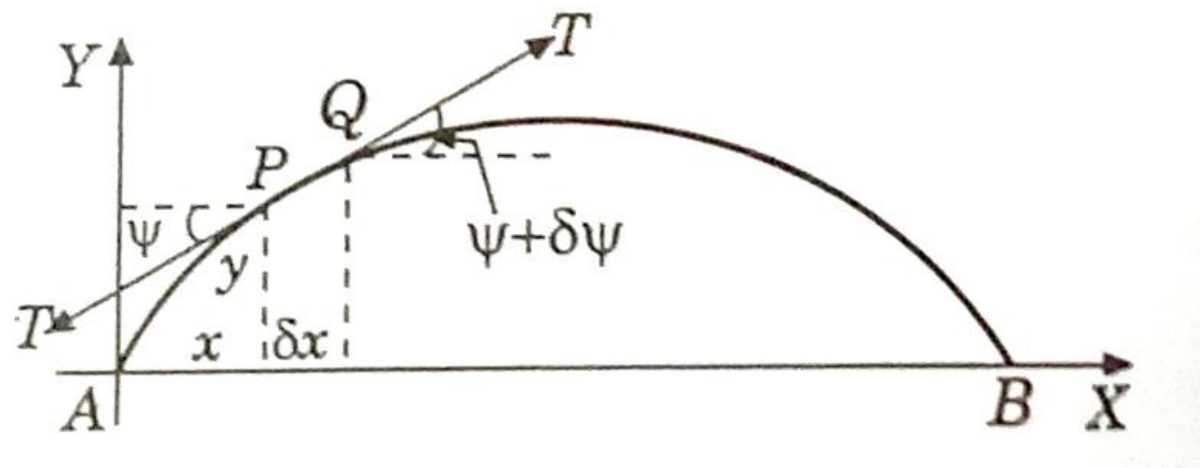
Applications of Partial differential equation

VIBRATIONS OF A STRETCHED STRING-WAVE EQUATION

Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension T. The tension T will be considered to be large as compared to theweight of the string so that the effects of gravity are negligible.

Let the string be released from rest and allowed to vibrate. We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.
Taking the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis; so that the motion takes place entirely in the xy-plane. The figure shows the string in the position APB at time t.

Consider the motion of the element PQ of the string between its points P (x, y) andQ($x + \delta x, y + \delta y$), where the tangents make angles Ψ and $\Psi + \delta\Psi$ with the x-axis. Clearly the element is moving upwards with the acceleration $\frac{\partial^2 y}{\partial t^2}$ and also the vertical component of the force acting on this element.



$$\sum F = T\sin(\Psi + \delta\Psi) - T\sin\Psi$$

$$= T[\sin(\Psi + \delta\Psi) - \sin\Psi]$$

$$= T[\tan(\Psi + \delta\Psi) - \tan\Psi] \qquad , since\ \Psi\ is\ small$$

$$= T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}\right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\delta x.\frac{\partial^2 y}{\partial t^2} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}\right]$$

$$i.e.\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}}{\delta x}\right]$$

Taking limits as $Q \to P\ i.e.\delta x \to 0$ we have

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \ , \qquad where\ c^2 = \frac{T}{m} \qquad (1)$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional wave equation.

Solution of wave equation.

Assume that a solution of (1) is of the form $y = X(x)T(t)$ where X is a function of x and T is a function of t only.

$$\frac{\partial^2 y}{\partial t^2} = X.T'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X''.T$$

$$XT'' = c^2 X'' T$$

$$i.e. \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \qquad (2)$$

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold well if each side is equal to a constant k (Say).

$$\frac{X''}{X} = k \text{ and } k = \frac{1}{c^2}\frac{T''}{T}$$

$$X'' = kX \quad \text{ and } T'' = kc^2 T$$

Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \qquad (3)$$

$$and\ \frac{d^2T}{dt^2} - kc^2 T = 0 \qquad (4)$$

Solving (3) and (4), we get

(i) When k is positive and $= p^2$,

Say $X = c_1 e^{px} + c_2 e^{-px}$; $T = c_3 e^{cpt} + c_4 e^{-cpt}$;

(ii) When k is negative and $= -p^2$,

Say

$$X = c_5 \cos(px) + c_6 \sin(px) \ ; T = c_7 \cos(cpt) + c_8 \sin(cpt)$$

(iii) When k is zero,

Say $X = c_9 x + c_{10}; T = c_{11} t + c_{12}$

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt}) \qquad (5)$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \qquad (6)$$

$$y = (c_9 x + c_{10})(c_{11} t + c_{12}) \qquad (7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be periodic function of x and t. Hence their solution must involve trigonometric terms.

Accordingly the solution given by (6), i.e., of the form

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

is the only suitable solution of the wave equation.

CHAPTER FIVE

CONCLUSION

Project was done on the application of the differential equations

In this project we have seen the applications of first order, second order and partial differential equation in the field of science and their importance in solving practical problems

We studied that the differential equation consists of three phases:

- Formulation of differential equation from the given physical situation, called modeling
- Solutions of this differential equation, evaluating the arbitrary constants from the given conditions, and
- Physical interpretation of the solution

We learned how first order differential equations

- Govern the motion of the object and boat.
- Its importance in the study of the Newton's law of cooling and its relevance in carbon dating.
- are used to find the concentration of the pollutant

We have seen how second order differential equations are used to study the behavior of motion of the spring under damped and un damped conditions and its relevance in LCR circuits

Finally we observed that the partial differential equations are used to govern the motion of wave.

We conclude from the project that differential equations have a significant role in the field of science.

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