A STUDY ON GRAPH COLORING AND ITS APPLICATIONS

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In partial fulfilment of the requirement for the award of
BACHERLOR OF SCIENCE IN MATHEMATICS

2017-2020

DEPARTMENT OF MATHEMATICS

ST. PAUL'S COLLEGE, KALAMASSERY

(AFFILIATED TO M G UNIVERSITY, KOTTAYAM)

CERTIFICATE

This is to certify that the project report titled "A STUDY ON GRAPH COLORING AND ITS APPLICATIONS" submitted by SURYA MOL K S(Reg no. 170021032270), JIPSON JAMES(Reg no. 170021032413) and LAYA RANSOM(Reg. no:170021032418) towards partial fulfilment of the requirements for the award of Degree of Bachelor of Science in Mathematics is a bonafide work carried out by them during the academic year 2017-2020.

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DECLARATION

We ,SURYA MOL K S (Reg. no:170021032270), JIPSON JAMES(Reg. no:170021032413) and LAYA RANSOM(Reg. no:170021032418) hereby declare that this project entitled "A STUDY ON GRAPH COLORING AND ITS APPLICATIONS" is an original work done by us under the supervision and guidance of prof. Valentine D'Cruz, faculty, Department of Mathematics in St. Paul's college Kalamassery in partial fulfilment for the award of The Degree of Bachelor of Science in Mathematics under Mahatma Gandhi University. We further declare that this project is not partly or wholly submitted for any other purpose and the data included in the project is collected from various sources and are true to the best of our knowledge.

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ACKNOWLEDGEMENT

For any accomplishment or achievement, the prime requisite is the blessing of the Almighty and it's the same that made this world possible. We bow to the lord with a grateful heart and prayerful mind.

It is with great pleasure that we express our sincere gratitude to our beloved teacher prof. Valentine D'Cruz, Department of Mathematics, St. Paul's College, for her overwhelming support, motivation and encouragement.

We would like to acknowledge our deep sense of gratitude to Dr. Savitha K S, Head of Department of Mathematics and all the faculty members of the department and our friends who helped us directly and indirectly through their valuable suggestions and self-criticisms, which came a long way in ensuring that this project becomes a success.

We also express our sincere gratitude to Prof. Valentine D'Cruz, Principal, St. Paul's College, Kalamassery for the support and inspiration rendered to us in this project report.

CONTENTS

INTRODUCTION

Graph theory was born in the 18^{th} century when the Swiss mathematician Leonard Euler, considered the problem of seven Konigsberg Bridge. Graph theory is the study of graphs, which are mathematical structure used to model pair wise relation between objects. A graph is the concept is made up of vertices, nodes or points which are connected by edges, arcs or lines.

In graph theory, graph coloring is a special case of labeling, it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph sch that no two adjacent vertices are of the same color, this is called vertex coloring. Similarly, an edge coloring assigns a color to each edge so that no two adjacent edges are of the same color and a face coloring of a planar graph assigns a color to each face or region so that no two faces that share a boundary have the same color.

Map coloring is an act of assigning different colors to different features on a map. In mathematics where the problem is to determine the minimum number of colors needed to color a map so that no two adjacent features have the same color.

Graph coloring problem is to assign colors to certain elements of a graph subject to certain constraints. It has a wide range of application, some of them are register allocation, scheduling, job scheduling, aircraft scheduling, time table scheduling and so on.

In this project, the opening chapter provides base knowledge of graph theory. This is followed by chapters vertex coloring, edge coloring, map coloring and the last chapter talk about the applications

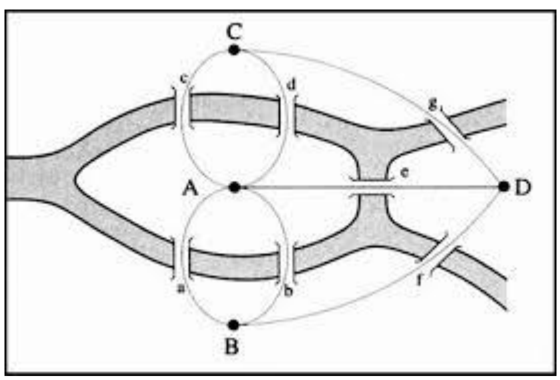
CHAPTER 1

PRELIMINARIES

1. Konigsberg and development of graph theory

Graph and graph theory began in the early 18th century when the Swiss mathematician, Leonard Euler considered the problem of seven Konigsberg Bridges.

The city of Konigsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the pregel river, and included two large islands - Kneiphof and Lomse- which were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of these bridges once and only once.



It is said that the townsfolk of Konigsberg amused themselves by trying to find a route that crossed each bridge just once. Euler considered this problem by using the graph given above in the figure, where each edge represents one of the seven bridges. He then showed the impossibility of such route by in effect showing that the graph has no Euler trail.

1.1 Graph

A graph G=(V(G), E(G)) consist of two finite sets V(G), the vertex set of the graph , often denoted by just V, which is a non-empty set of elements called vertices, and E(G), the edge set of the graph, often denoted by just E, which is possibly a empty set of elements called edges.

1.2 Complete Graph

A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge.

1.3 Bipartite Graph

Let G be a graph. If the vertex set f G can be partitioned into two non-empty subsets X and Y (that is, XUY=V and X∩Y=ϕ) in such a way that each edge of G has one end in X and other end in Y, then G is called bipartite. The partition V=XUY is called a bipartition of G.

1.4 Complete Bipartite Graph

A complete bipartite graph is a simple bipartite graph G, with bipartition V=XUY, in which every vertex in X is joined to every vertex of Y. If X has m vertices and Y has n vertices, such a graph is denoted by $k_{m,n}$.

1.5 Incident Vertex

An edge e of a graph G is said to be incident with the vertex v, if v is an end vertex of e. In this case we also say that v is incident with e.

1.6 Adjacency

Two edges are adjacent if they are incident with a common vertex.

1.7 Vertex Degree

Let v be a vertex of graph G. The degree d(v) (or $d_G(v)$) of v is the number of edges of G incident with v.

1.7 Sub graph

Let H be a graph with vertex set V(H) and edge set E(H) and similarly, let G be a graph with vertex set V(G) and E(G). Then we say that H is a sub graph of G if V(H) is a subset of V(G) and E(H) is a subset of E(G). In such a case, we also say that G is a super graph of H.

1.8 Path and Cycles

1.8.1 Walk

A walk in a graph is a finite sequence $W=ve_1v_1e_2\ldots e_kv_k$ whose terms are alternatively vertices and edges such that each edge e_1 has end vertices v_{k-1} and v_i. 'W' is called v-v_k walk. The vertex v is called origin and v_k is called terminal of walk W. The vertices $v_1,v_2,v_3,\ldots v_k$ are called internal vertices of W. The integer k, the number of edges in walk is called the length of the walk W.

1.8.2 Closed Walk

Given two vertices u and v of a graph G, a u-v walk is called closed or open depending on whether u=v implies that the walk is closed and u≠v implies that the walk is open.

1.8.3 Trail

If the edges $e_1,e_2,e_3,\ldots,e_k$ of the walk, $W=v_0e_1v_1e_2\ldots e_kv_k$ are distinct then w is called a trail. A trail is a walk with no edges is repeated.

1.8.4 Path

If vertices $v_0,v_1,v_2,\ldots,v_k$ of the walk $W=v_0e_1v_1e_2\ldots e_kv_k$ are distinct then W is called a path. A path with n vertices will be denoted by p_n and has length (n-1).

1.8.5 Cycle

A non-trivial closed trail in a graph G is called a cycle if its origin and internal vertices are distinct. The closed trail $C=v_1v_2v_3\ldots.v_nv_1$ is a cycle, if C has at least one edge and $v_1v_2\ldots.v_n$ are n distinct vertices. A cycle of length k (with k edge) is called a k-cycle. An n- cycle is denoted by C_n with n vertices.

1.10 Connected Graph

A vertex u is said to be a connected to a vertex v in a graph G if there is a path in G from u to v. A graph G is connected if every two vertices are connected.

Disconnected Graph

A graph G that is not connected is called a disconnected graph.

1.11 Bridge (cut edge)

An edge e of a graph G is called a (a cut edge or an isthmus) if the subgraph G-e has more connected component than G has.

1.12 Cut Vertex

A vertex v of a graph G is called a cut vertex (or a articulation point) of G if G-v has more connected components than G has.

CHAPTER – 2

<u>VERTEX COLORING</u>

Colors can be used in graphs to model problems where one wishes to avoid some form of "Interference" or ensure some "Independence".

<u>Definition 2.1</u>:

A *vertex coloring* of a graph G is a mapping $f: V(G) \rightarrow S$, where S is a set of distinct color's that assigns colors to vertices such that adjacent vertices are assigned different colors; it is *proper* if no two adjacent vertices receive the same color. Thus, a *proper vertex coloring* f of G is a function $f: V(G) \rightarrow S$ such that $f(u) \neq f(v)$, whenever vertex u and v are adjacent in G.

A *k-coloring* of G is a coloring which consists of k different colors and G is said to be *k-colorable*.

E.g: In the Lame Duck Airline problem, Frank Drake, the owner of Lame Duck, wants the flights to take place only on Mondays, Wednesdays and Fridays. He also wants no more than one flight per day visiting any of the towns. To see if this is possible, he constructs the graph of Figure *2.1* which has seven vertices, one representing each of the proposed flights, and where an edge joins two vertices if the corresponding flights have a town in common.

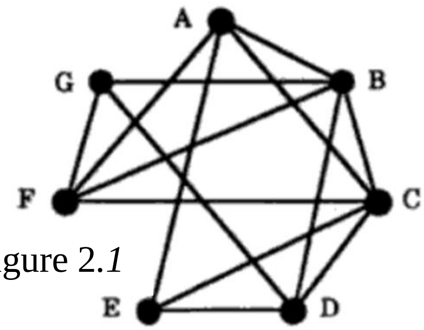


Figure 2.*1*

The Lame Duck Airline network.

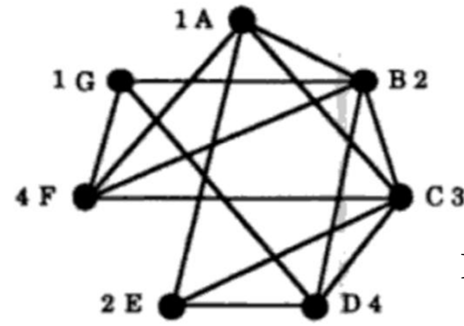


Figure 2.*2*

The Lame Duck flies again.

The question being asked amounts to whether or not the graph of Figure *2.1* has a *3*-coloring. But clearly, Figure *2.2* shows that it has a *4*-coloring.

Definition 2.2:

The minimum number *n* for which there is an *n*-coloring of the graph *G* is called the *Chromatic index* or *Chromatic number* of *G* and is denoted by $\chi(G)$. If $\chi(G) =$ k, we say that *G* is *k-chromatic*.

In the Lame Duck Airlines problem, since we have displayed a *4*-coloring of *G* (Figure *2.2*), this means that $\chi(G) = 4$. Thus the seven flights can be scheduled on four days but not three, subject to the stated restrictions.

For example, the *chromatic number* of:

a) The *Petersen graph* is $\chi(G) = 3$ (Figure *2.3*).

b) The *Grötzsch graph* is $\chi(G) = 4$ (Figure *2.4*).

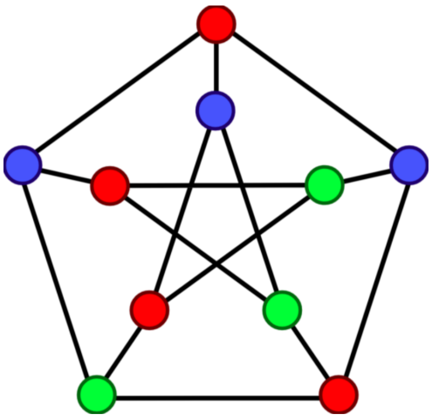


Figure 2.3: $\chi(G) = 3$.

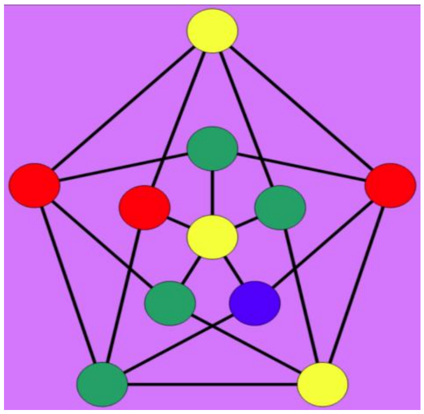


Figure 2.4: $\chi(G) = 4$.

Note 1:

If the graph G has a loop at the vertex v, then v is *adjacent* to itself and so no coloring of G is possible. So we will assume that in any vertex coloring, content graphs have no loops.

Note 2:

Two distinct vertices are *adjacent*, if there is at least one edge between them. So for our purpose, all but one of the set of parallel edges may be ignored. That is, the graphs using for vertex coloring are simple graphs.

Theorem 2.1:

a) If the graph G has n vertices, then $\chi(G) \leq n$.

b) If H is subgraph of the graph G, then $\chi(H) \leq \chi(G)$.

c) $\chi(K_n) = n$ for all $n \geq 1$.

d) If the graph G contains K_n as a subgraph, then $\chi(G) \geq n$.

e) If the graph G has $G_1, G_2, \ldots, G_n$ as its connected components, then

$$\chi(G) = \max_{1 \leq i \leq n} \chi(G_i).$$

Proof:

From theorem *2.1*(d), we can now solve the Lame Duck Airlines problem. The problem graph in Figure *2.1* has K_4 as a subgraph (induced by the vertices *A, B, C* and *F*) and so $\chi(G) \geq 4$. Since we have a *4*-coloring of *G*, this means $\chi(G) = 4$. Thus, the seven flights can be scheduled on four days but not three.

Let us now look at some simple examples. Firstly, if the graph *G* has no edges, then each vertex can be given the same color, i.e., $\chi(G)= 1$. Clearly, the converse also holds. Thus $\chi(G) = 1$ if and only if *G* is an empty graph.

Now let $G = C_n$, the cycle of length *n*, with vertices $V_1, V_2, \ldots, V_n$ appearing in order around the cycle. If we assign color *1* to V_1, V_2 must be colored differently, say, by color *2*. But then, we may color V_3 by *1* again. Continuing in this fashion round the cycle, we see that if *n* is odd, then V_n needs a different color, say, color *3*. Thus $\chi(C_n) = 2$ if *n* is even and *3* if *n* is odd.

Theorem 2.2:

Let *G* be a non-empty graph. Then $\chi(G)=2$ if and only if *G* is bipartite.

Proof:

Let *G* be bipartite with bipartition $V=X \cup Y$. Assigning color *1* to all vertices in *X* and color *2* to all vertices in *Y*, gives *2*-coloring for *G* and so, since *G* is non-empty, $\chi(G) = 2$.

Conversely, suppose that $\chi(G) = 2$. Then *G* has *2*-coloring. Denoted by *X*, the set of all those vertices colored *1* and by *Y*, the set of all vertices colored *2*. Then no two vertices in *X* are adjacent and similarly for *Y*. Thus any edge in *G* must join a vertex in *X* and a vertex in *Y*. Hence, *G* is bipartite with bipartition $V = X \cup Y$.

Corollary:

Let *G* be a graph. Then, $\chi(G) \geq 3$, if and only if *G* has an odd cycle.

Proof:

Unlike the $n = 2$ case, there is no easy characterization of graphs with chromatic index 3, or for that matter, higher index of an arbitrary graph G, provided the degree of all their vertices of G. For this, first we need some notion.

Note 3:

For a graph G, we let $\Delta(G) = max \{d(v) : v \text{ is a vertex of } G\}$.

Thus $\Delta(G)$ is the maximum vertex degree of G.

Theorem 2.3:

For any graph G, $\chi(G) \leq \Delta(G) + 1$.

Proof:

To prove, we use induction on n, the number of vertices in G. Since, here $G = k_1$, $\chi(G) = 1$ and $\Delta(G) = 0$. The theorem is true for $n = 1$.

Now suppose that the result is true for all graphs with $n – 1$ vertices. When n is a fixed vertex of G, the subgraph $G – V$ has $n – 1$ vertices and so by the induction assumption, $\chi(G – V) \leq \Delta(G – V) + 1$. This allows us to choose a vertex coloring of $G – V$ involving $\Delta(G – V) + 1$ colors. Now, our vertex v has at most $\Delta(G)$ neighbors and we can use such a color for v, giving a $\Delta(G + 1)$ coloring for G.

On the other hand, if $\Delta(G) \neq \Delta(G – V)$ then, $\Delta(G – V) < \Delta(G)$ and simply coloring v with a brand new color, gives a $\Delta(G – V + 2)$-coloring of G, which is good enough since $\Delta(G – V) + 2 \leq \Delta(G) + 1$. Hence in both cases, we have $\chi(G) \leq \Delta(G) + 1$. Hence, the proof.

If $G = k_n$ or a cycle of odd length, we actually have $\chi(G) = \Delta(G) + 1$. However, we can often improve upon theorem. For this purpose, we describe a technique which allows us in certain.

CHAPTER – 3

EDGE COLOURING

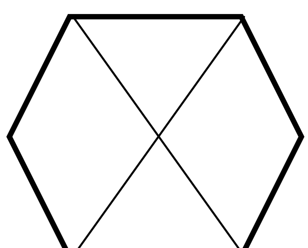
Definition:

An *edge colouring* of a loopless graph G is a function Π: E(G)→ S, where S is a set of distinct colours; it is proper if no two adjacent edges receive the same colour. Thus a proper edge-colouring of G is a function Π: E(G) → S such that Π(e) ≠ Π(e') whenever edge e and e' are adjacent in G.

Let G be a empty graph. An edge coloring of G assigns colors, usually denoted by 1,2,3,…to the edges of G, one color per edge, so that adjacent edges are assigned different colors.

A K-edge coloring of G is a coloring of G which consists of K different colors and in this case G is said to be K- edge colorable.

The minimum number n for which there is an n-coloring of G is called the Edge chromatic number or Edge chromatic index of G and is denoted by $X_1(G)$.If $X_1(G)=k$, we say that it is K-edge chromatic.



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3.1 ELEMENTARY PROPERTIES

(1) If H is a subgraph of G, then $X_1(H) \leq X_1(G)$

(2) Let $\Delta(G)$ denote the maximum vertex degree of G as usual, we have $\Delta(G) \leq X_1(G)$.Since if v is any vertex of G with $d(v) = \Delta(G)$,then the $\Delta(G)$edges incident with v must have a different colour in any edge coloring of G.

$X_1(G)$ is either $\Delta(G)$ OR $\Delta(G)+1$

3.2 KEMPE CHAIN ARGUMENT

Let G be a graph with an edge coloring involving at least 2 different colors, denoted by i & j. Let H(i,j) denote the subgraph of G induced by all the edges colored either i or j. Let K be a connected component of this subgraph. Then, as the reader can easily check, K is just a path whose edges are alternatively colored by i& j and if we interchange the colors on these edges, but leave the colors on all the other edges of G unchanged, the result is a new coloring of G, involving the same initial colors. As in the vertex coloring situation we refer to such a component k as a Kempe chain and this recoloring technique as the Kempe chain argument.

Given an edge coloring of the graph G involving the color we say that it is present at a vertex v of G if there is an edge colored incident with v. If there is no such edge incident with v then we say that i is absent from v.

Theorem 3.3:

Let *G* be a non-empty bipartite graph. Then $\chi'(G) = \Delta(G)$.

Proof:

The proof is by induction on the number of edges of *G*. The result is clearly true if *G* has just one edge. Now let *G* have more than one edge and assume that the result is true for all non-empty bipartite graphs with fewer edges than *G*. Since $\Delta G \leq$

$\chi'(G)$ is suffices to prove that G has a $\Delta(G)$-edge coloring. To simplify the notation let $\Delta(G) = k$. Let e be some fixed edge of G. Then the edge-deleted subgraph $G - e$ is bipartite with less edges than G and so, by the induction assumption, has a $\Delta(G - e)$-edge coloring and is a k-coloring, since $\Delta(G - e) \leq \Delta(G) = k$. We will show that the same k colors can be used to color G.

Let the uncolored edge e have vertices u and v. Since $d(u) \leq k$ in G and e is uncolored, there is at least one of the k colors absent from u. Similarly, at least one of these colors is absent from v. If there is a color absent from both u and v, simply use it to color e and we get a k-edge coloring of G, as required. Thus we are left to deal with the case where there is a color i present at u but absent from v and a color j present at v but absent from u.

Let K be the Kempe chain containing u in the subgraph $H(i,j)$ induced by the edges colored i or j. Now suppose that v is also in K. Then, there is a path P in K from u and v. Since u and v are adjacent, they do not belong to the same bipartition subset of the bipartite graph G and so, the path P must have odd length. Moreover, since the color i is present at u, the first edge of P is colored i. Since the edges of P are alternately colored i and j, and P is of odd length, this implies that the last edge of P, that incident with v, is also colored i. This is a contradiction since i is absent from v. Hence v does not belong to the Kempe chain K.

We now use the Kempe chain argument on K. This interchanging of colors makes i now absent from u, but does not affect the colors of the edges incident with v. Thus i is absent from both u and v in this new k-edge coloring. As before, now we can safely color edge e by i to produce a k-edge coloring of G.

Before we go on to look at the edge-chromatic index of complete graphs, let us briefly describe an application of this theorem to the construction of Latin squares.

A *Latin square* (of order n) is an $n \times n$ matrix having the numbers $1, 2, \ldots, n$ as entries such that no single number appears in more than one row and in more than in one column. They are used frequently by statisticians and quality control analysts in experimental designs. We can show that a Latin square of order n can be constructed using an n-edge coloring of the complete bipartite graph $K_{n,n}$. Note that $\Delta(K_{n,n}) = n$ and so by this theorem, $K_{n,n}$ does have an n-edge coloring but no edge coloring with less than n colors.

Theorem 3.4

Let G=Ka, the complete graph on n vertices, n≥2.Then $X_1(G)$={ Δ(G)(=n-1) if n is even

Δ(G)+1(0=) if n is odd}

Theorem 3.5 (Vizing's Theorem):

Although it is true that for any loopless graph G, $\chi'(G) \geq \Delta(G)$, it turns out that for any simple graph G, $\chi'(G) \leq 1 + \Delta(G)$. This major result in edge coloring of graphs was established by Vizing and independently by Gupta.

That is, the theorem states that, for any simple graph G,

$$\Delta(G) \leq \chi'(G) \leq 1 + \Delta(G).$$

Theorem 3.6

Let G be a nontrivial graph .Then $\Delta(G) \leq X_1(G) \leq \Delta(G)+1$

CHAPTER 4

MAP COLORING

Definition 4.1:

A *map* is defined to be a plane connected graph with no bridges**.**

Definition 4.2:

A map *G* is said to be *k-face colorable***,** if we can color its faces with at most *k* colors in such a way that no two adjacent faces, i.e., two faces sharing a common boundary edge, have the same colors.

Definition 4.3:

The *Four Color Conjecture* states that, if the plane is divided into regions and the regions are colored such that no two regions with a common edge have the same color, then at most four colors are required, i.e., every map is *4*-face colorable.

THE FOUR CONJUCTURE: Every map is 4 colorable.

Theorem 4.1

(a) A map G is K-face colorable if an only if its dual G is k-vertex colorable

(b) Let G be a plane connected graph without loops , then G has a vertex coloring of k colors if and only if its dual G* has a k-face coloring.

Theorem 4.2

A map G is two face colorable if and only if its an Euler graph.

Theorem 4.3

Let G be a cubic map, ie a map in which each vertex has degree 3. Then G has as 3-face coloring if and only if each of its faces has even number of edges on its boundary.

Proof

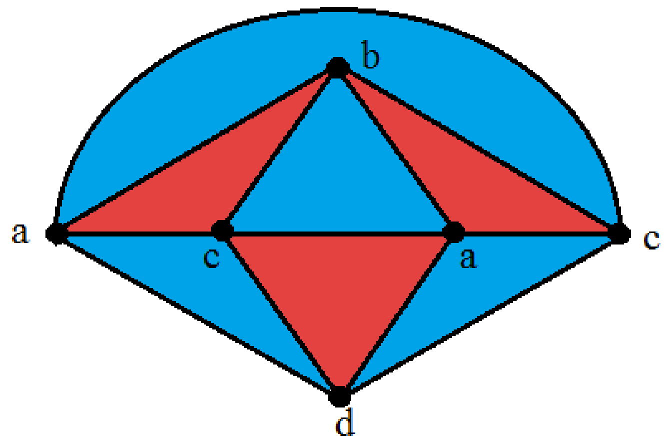
First suppose that G has a 3-face coloring using colors α, β and γ.

Let 'f' be any interior face of G colored α. Then the faces surrounding f must be colored β or γ. Looking at these faces in turn as we go clockwise round f, since no 2 faces of the same color can be adjacent, those colored β must alternate with those colored γ and they must be even in number. Since each of these faces corresponds to as edge on the boundary of f, it follows that f has even degree. A similar argument applies to the interior face of G.

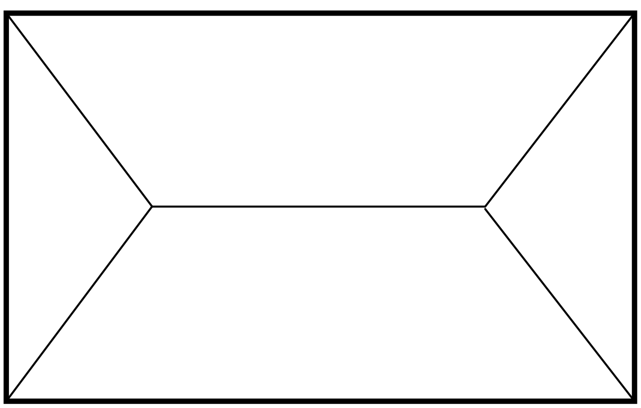
Conversely, we first prove a dual result. Let H be a plane connected Euler graph in which every face has degree three, ie, each face is triangle. We will show that H has a 3-colouring,ie,a vertex coloring of three colors. First, its straight forward to see that every edge of H is a part of a cycle and so H has no bridge. Hence H is Euler map and so H has no bridge. Hence H is Euler map and so, by theorem 4.2, H is 2-face colorable. Let us choose red and blue to color the face of H.

Now choose a red face f of H. Starting at a particular vertex of f and visiting the other 2 traveling clockwise, color the first vertex a, the second vertex b and third c. Any face g adjacent to f is colored. Color the remaining vertex of G with the third unused color. This result in the three colors a, b and c being assigned, in that order, to the vertices of G in anticlockwise fashion.

We can now extend this vertex coloring to all the vertices of H, resulting in a clockwise allocation of a,b, and c to red faces and an anticlockwise allocation to blue faces, as shown in figure. Thus we have shown that if H is any plane connected Euler graph in which every face has degree three, then H has 3-colouring



Now let G be a cubic map in which each face has even degree. Then In the dual map G* each vertex has even degree while each face has degree three. Hence by our arguments of the previous two paragraphs G* has a 3-colouring. Thus, by theorem 4.1, that not all maps are 3-face colorable. For example, the cubic map of figure has faces of odd degree and so by the theorem, cannot be 3-face colorable.



Of course the 4 color says that just one more color is needed to be able to color all maps. We will prove shortly the five colors are sufficient but first we give an easy proof that six color suffice.

Definition

A plane graph in which every face has degree three is called a triangulation.

Theorem

THE FOUR COLOUR THEOREM

Every map can be colored in four or fewer colors .

In mathematics , the four color theorem, or the four color map theorem, states that given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color. Adjacent means that two regions share a common boundary curve segment, not merely a corner where three or more regions meet. It was the first major theorem to be proved using a computer. Initially this proof was not accepted by all mathematicians because the computer assisted proof was infeasible for a human to check by hand. Since then the proof has gained wide acceptance, although some doubters remain.

CHAPTER – 5

APPLICATIONS OF GRAPH COLORING

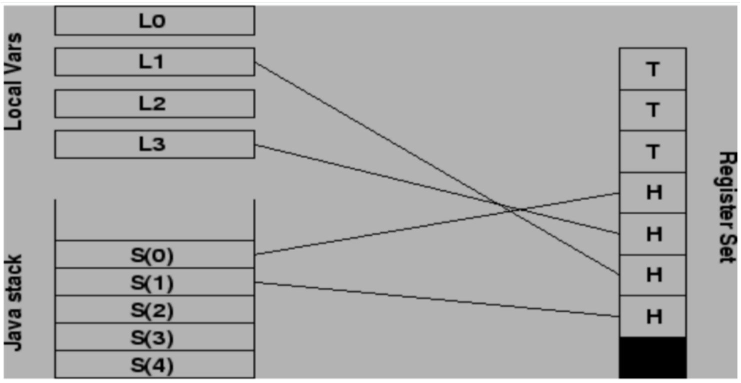
1.Scheduling

Vertex coloring models to a number of scheduling problems. In the cleanest form, a given set of jobs need to be assigned to time slots, each job requires one such slot. Jobs can be scheduling in any order, but pairs of jobs may be in conflict in the sense that they may not be assigned to the same time slot, for example because they both rely on a shared resource. The corresponding graph contains a vertex for every job and an edge for conflicting pairs of jobs. The chromatic number of the graph is exactly the minimum make span, the optimal time to finish all jobs without conflicts.

Details of the scheduling problem define the structure of the graph. For example, when assigning aircraft to flights, the resulting conflicting graph is an interval graph, so the coloring problem can be solved efficiently. In bandwidth allocation to radio stations, the resulting conflict graph is a unit disk graph, so the coloring problem is 3-approximable.

2.Register Allocation

A compiler is a computer program that translates one computer language into another. To improve the execution time of the resulting code, one of the techniques of compiler optimization is register allocation, where the most frequently used values of the complied program are kept in the fast processor registers. Ideally, values are assigned to registers so that they can all reside in the registers when they are used.



In compiler optimization, register allocation is the process of assigning a large number of target program variables onto a small number of CPU registers. This problem is also a graph coloring problem. Ideally, values are assigned to registers so that they can all reside in the registers when they are used. This is a model of a graph coloring problem, where the compiler constructs an *interference graph G* of the program and vertices are variables. An edge connects two vertices if they are needed at the same time. If the graph can be colored with *k* colors, then any set of variables needed at the same time can be stored in at most *k* registers and the uncolored variables are split into memory.

3.GSM Mobile Phone Networks:

Groups Special Mobile (GSM) is a mobile phone network created to provide a standard for a mobile telephone system, where the geographical area of this cellular network is divided into hexagonal regions or cells. Each cell has a communication tower which connects with mobile phones within the cell. All mobile phones connect to the GSM network by searching for cells in the immediate vicinity. Since GSM operate only in four different frequency ranges, it is clear by the concept of graph theory that only four colors can be used to color the cellular regions. These four different colors are used for proper coloring of the regions. Therefore, the vertex coloring algorithm may be used to assign at most four different frequencies for any GSM mobile phone network.

For a map drawn on the plane or on the surface of a sphere, the *Four Color Theorem* asserts that it is always possible to color the regions of a map properly using at most four distinct colors such that no two adjacent regions are assigned the same color. Now, a dual graph is constructed by putting a vertex inside each region of the map. Connect two distinct vertices by an edge if their respective regions share a whole segment of their boundaries in common. Then proper coloring of the dual graph gives proper coloring of the original map. Since, coloring the regions of a planar graph G is equivalent to coloring the vertices of the dual graph and vice versa, by coloring the map regions using *Four Color Theorem*, the four frequencies can be assigned to the regions accordingly.

4. Sudoku:

Sudoku is a very popular puzzle. The puzzle consists of a *9x9* grid with digits so that each column, each row and each of the nine *3x3* sub-grids that compose the grid contains all of the digits from *1* to *9* appearing once, i.e., numbers in rows are not repeated, numbers in columns are not repeated and numbers in *3x3* sub-grids are not repeated (order of the number when filling is not important). This can be viewed as graph coloring.

			2	6		7		1
6	8			7			9	
1	9				4	5		
8	2		1				4	
		4	6		2	9		
	5				3		2	8
		9	3				7	4
	4			5			3	6
7		3		1	8			

4	3	5	2	6	9	7	8	1
6	8	2	5	7	1	4	9	3
1	9	7	8	3	4	5	6	2
8	2	6	1	9	5	3	4	7
3	7	4	6	8	2	9	1	5
9	5	1	7	4	3	6	2	8
5	1	9	3	2	6	8	7	4
2	4	8	9	5	7	1	3	6
7	6	3	4	1	8	2	5	9

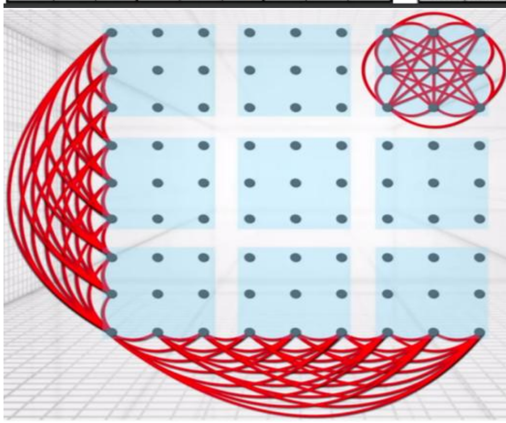


Figure *5.2*

In general, graph coloring is the assignment of colors to the vertices of a graph such that no two adjacent vertices have the same color. Here, the graph will have *81* vertices with each vertex corresponding to a cell in the grid. Two distinct vertices will be adjacent if and only if the corresponding cells in the grid are either in the same row, or same column, or same sub-grid. Each completed Sudoku square then corresponds to a k-coloring of the graph. Connect every pair of vertices whose squares are buddies by edge. Then each vertex connects to *20* other vertices, i.e., $81x20/2 = 810$ edges. This is same as finding *9*-coloring graph (Figure *5.2*).

5. Aircraft scheduling:

Assume that there are k aircrafts and they have to be assigned to n flights, where the i-th flight should be during the time interval $(a_i\ ,b_i)$. Clearly, if two flights overlap, then the same aircraft cannot be assigned to both the flights. This problem is modeled as a graph as follows.

Let the vertices represent airports. Consider there is an edge from vertex A to vertex B, if there is a direct flight from the airport represented by A to the airport represented by B. Airlines use minimum spanning trees to work out their basic route system. Now, the vertices of the conflict graph correspond to the flights. Two vertices will be connected, if the corresponding time intervals overlap. Therefore, the graph is an interval graph that can be colored optimally in polynomial time.

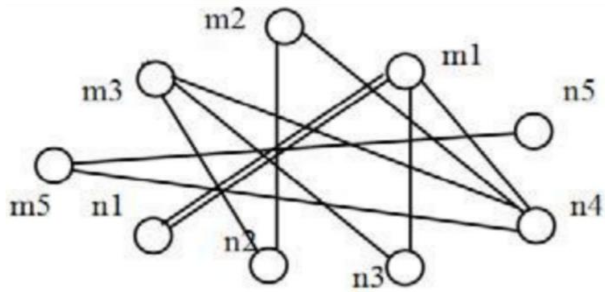
6. Minimum sum coloring:

In minimum sum coloring, the sum of the colors assigned to the vertices is minimal in the conflict graph. The minimum sum coloring technique can be applied to the scheduling theory of minimizing the sum of completion times of the jobs, which is the same as minimizing the average completion time. The multicoloring version of the problem can be used to model jobs with arbitrary lengths. Here, the finish time of a vertex is the largest color assigned to it and the sum of coloring is the sum of the finish time of the vertices. That is, the sum of the finish times in a multicoloring is equal to the sum of completion times in the corresponding schedule.

7. Time table scheduling:

Allocation of classes and subjects/periods to the professors is one of the major issues if the constraints are complex and graph theory (particularly, graph coloring) plays an important role in this problem. Suppose in a university, there are m professors $T_1, T_2, \ldots, T_m$ and n classes $C_1, C_2, \ldots, C_n$. Each professor T_i is expected to teach the class C_j for p_{ij} periods. It is clear that during any particular period, no more than one professor can handle a particular class and no more than one class can be engaged by any professor. Our aim is to schedule a complete timetable for the day with the minimum possible number of periods. This problem is known as the ‘Timetable Problem’.

We represent this problem by a bipartite graph G with bipartition (T,C), where T represents the set of professors T_i and C represents the set of classes C_j. Further, T_i is made adjacent to C_j in G with p_{ij} parallel edges if and only if professor T_i is to handle class C_j for p_{ij} periods. Now in any one period, it is presumed that each professor can teach at most one class, and each class can be taught by maximum one professor. We color the edges of G so that no two adjacent edges receive the same color. Then edges in a particular color class, i.e., the edges in that color, forms a matching in G and corresponds to a schedule of work for a particular period. Hence, the minimum number of periods required is the minimum number of colors in an edge coloring of G, in which adjacent edges receive distinct colors. In other words, it is the edge-chromatic number of G. Thus a teaching schedule for one period corresponds to a matching in the graph and conversely, each matching corresponds to a possible assignment of professors to classes for one period. Hence, if no professor teaches for more than p-periods, then the teaching requirements can be scheduled in a p-period timetable. We thus have a complete solution to the problem, i.e., it is equal to the chromatic number of the graph. For example, consider there are 4 professors, namely m_1, m_2, m_3 and m_4, and 5 subjects, say n_1, n_2, n_3, n_4 and n_5 to be taught.



p	n1	n2	n3	n4	n5
m1	2	0	1	1	0
m2	0	1	0	1	0
m3	0	1	1	1	0
m4	0	0	0	1	1

(a) Figure *5.3* (b)

Then the bipartite graph (Figure *5.3(a)*) and its teaching requirement matrix $p = [p_{ij}]$ (Figure *5.3(b)*) with *4* professors and *5* subjects is as shown.

The proper coloring of the graph can be done by four colors using the vertex coloring algorithm, which leads to the edge coloring of the bipartite multigraph *G*. In a similar manner, graph coloring can be used to schedule exams so that no two exams with a con student are scheduled at the same time.

8. Job scheduling:

Here the jobs are assumed as the vertices of the graph and there is an edge between two jobs, if they cannot be executed simultaneously. Then there is a *1-1 correspondence* between the feasible scheduling of the jobs and the colorings of the graph.

PRACTICAL EXERCISE OF MAP COLORING

The convention of using colors comes from coloring countries on a map where each country should have a different color from its neighbor. However, countries on a map is an example of a planar graph and for planar graphs, four colors are enough. In the case of non-planar graphs, we do not know how many colors are required.A geographical map of countries or states drawn on the plane or the surface of a sphere, where no two adjacent cities are assigned same color is an example of Graph coloring. This is particularly known as Map coloring and it is possible to color any map in four colors using *Four Color Theorem*.

Here is a practical example of Map coloring of an Indian state, Kerala, which has *14* districts, using four colors, where no two adjacent districts are assigned same color.

CONCLUSION

Graph coloring is still a very active field of research, we have studied some important theorems on vertex coloring, edge coloring in this project. Graph coloring enjoys many practical applications as well as theoretical challenges. Beside the classical types of problems, different limitations can also be set on the graph, or on the way a color is assigned, or even on the color itself. It has even reached popularity with the general public in the form of the popular number puzzle Sudoku.

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